Zadatak 201 (Srna, gimnazija)

Riješi jednadžbu: $\left(\frac{1}{2}\right)^{x} \cdot \left(\frac{2}{3}\right)^{x+1} \cdot \left(\frac{3}{4}\right)^{x+2} = 6.$

Rješenje 201

Ponovimo!

$$a^{n} \cdot a^{m} = a^{n+m} \quad , \quad a^{1} = a \quad , \quad a^{n} \cdot b^{n} = (a \cdot b)^{n} \quad , \quad a^{-n} = \frac{1}{a^{n}} \quad , \quad \left(a^{n}\right)^{m} = a^{n \cdot m}.$$

$$a^{f(x)} = a^{g(x)} \;\Rightarrow\; f(x) = g(x).$$

$$\left(\frac{1}{2}\right)^{x} \cdot \left(\frac{2}{3}\right)^{x+1} \cdot \left(\frac{3}{4}\right)^{x+2} = 6 \Rightarrow \left(\frac{1}{2}\right)^{x} \cdot \left(\frac{2}{3}\right)^{x} \cdot \left(\frac{2}{3}\right)^{1} \cdot \left(\frac{3}{4}\right)^{x} \cdot \left(\frac{3}{4}\right)^{2} = 6 \Rightarrow$$

$$\Rightarrow \left(\frac{1}{2}\right)^{x} \cdot \left(\frac{2}{3}\right)^{x} \cdot \frac{2}{3} \cdot \left(\frac{3}{4}\right)^{x} \cdot \frac{9}{16} = 6 \Rightarrow \left(\frac{1}{2}\right)^{x} \cdot \left(\frac{2}{3}\right)^{x} \cdot \left(\frac{3}{4}\right)^{x} \cdot \frac{2}{3} \cdot \frac{9}{16} = 6 \Rightarrow \left(\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4}\right)^{x} \cdot \frac{18}{48} = 6 \Rightarrow$$

$$\Rightarrow \left(\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4}\right)^{x} \cdot \frac{3}{8} = 6 \Rightarrow \left(\frac{1}{4}\right)^{x} \cdot \frac{3}{8} = 6 \Rightarrow \left(\frac{1}{4}\right)^{x} \cdot \frac{3}{8} = 6 \;/ \cdot \frac{8}{3} \Rightarrow \left(\frac{1}{4}\right)^{x} = 6 \cdot \frac{8}{3} \Rightarrow \left(\frac{1}{4}\right)^{x} = 6 \cdot \frac{8}{3} \Rightarrow$$

$$\Rightarrow \left(\frac{1}{4}\right)^{x} = 16 \Rightarrow \left(4^{-1}\right)^{x} = 4^{2} \Rightarrow 4^{-x} = 4^{2} \Rightarrow -x = 2 \Rightarrow -x = 2 \;/ \cdot (-1) \Rightarrow x = -2.$$

Vježba 201

Riješi jednadžbu: $\left(\frac{1}{2}\right)^{x} \cdot \left(\frac{2}{3}\right)^{x+1} \cdot \left(\frac{3}{4}\right)^{x+2} = 24.$

Rezultat: – 3.

Zadatak 202 (Srna, gimnazija)

Riješi jednadžbu: $a^{2 \cdot x-1} + a^{2 \cdot x+1} = a^{3} \cdot \left(1+a^{2}\right).$

Rješenje 202

Ponovimo!

$$a^{n} \cdot a^{m} = a^{n+m} \quad , \quad a^{1} = a \quad , \quad a^{-n} = \frac{1}{a^{n}} \quad , \quad a^{f(x)} = a^{g(x)} \;\Rightarrow\; f(x) = g(x).$$

Zakon distribucije množenja prema zbrajanju

$$a \cdot (b+c) = a \cdot b + a \cdot c \quad , \quad a \cdot b + a \cdot c = a \cdot (b+c).$$

$$a^{2 \cdot x-1} + a^{2 \cdot x+1} = a^{3} \cdot \left(1+a^{2}\right) \Rightarrow a^{2 \cdot x} \cdot a^{-1} + a^{2 \cdot x} \cdot a^{1} = a^{3} \cdot \left(1+a^{2}\right) \Rightarrow$$

$$\Rightarrow a^{2 \cdot x} \cdot \frac{1}{a} + a^{2 \cdot x} \cdot a = a^{3} \cdot \left(1+a^{2}\right) \Rightarrow a^{2 \cdot x} \cdot \left(\frac{1}{a} + a\right) = a^{3} \cdot \left(1+a^{2}\right) \Rightarrow$$

$$\Rightarrow a^{2 \cdot x} \cdot \frac{1+a^{2}}{a} = a^{3} \cdot \left(1+a^{2}\right) \Rightarrow a^{2 \cdot x} \cdot \frac{1+a^{2}}{a} = a^{3} \cdot \left(1+a^{2}\right) \;/ \cdot \frac{a}{1+a^{2}} \Rightarrow a^{2 \cdot x} = a^{4} \Rightarrow$$

$$\Rightarrow 2 \cdot x = 4 \Rightarrow 2 \cdot x = 4 \;/ : 2 \Rightarrow x = 2.$$

Vježba 202

Riješi jednadžbu: $a^{2 \cdot x-1} + a^{2 \cdot x+1} = a \cdot \left(1+a^{2}\right).$

Rezultat: x = 1.

Zadatak 203 (Ana, ekonomska škola)

Nađi vrijednost izraza: $\left(ctg\,\frac{\pi}{6}\right)^{6}+4^{-\log 0.01}.$

Rješenje 203

Ponovimo!

$$\left(\sqrt{a}\right)^{2\cdot n}=a^{n} \quad , \quad \log 0.01=-2.$$

$$\left(ctg\,\frac{\pi}{6}\right)^{6}+4^{-\log 0.01}=\left(\sqrt{3}\right)^{6}+4^{-(-2)}=3^{3}+4^{2}=27+16=43.$$

Vježba 203

Nađi vrijednost izraza: $\left(ctg\,\frac{\pi}{6}\right)^{6}+4^{-\log 0.1}.$

Rezultat: 31.

Zadatak 204 (Ana, ekonomska škola)

Riješi jednadžbu: $\frac{3^{x}-3^{-x}}{3^{x}+3^{-x}}=\frac{4}{5}.$

Rješenje 204

Ponovimo!

$$a^{n}\cdot a^{m}=a^{n+m} \quad , \quad a^{0}=1 \quad , \quad a^{1}=a \quad , \quad a^{-n}=\frac{1}{a^{n}} \quad , \quad a^{f(x)}=a^{g(x)} \Rightarrow f(x)=g(x).$$

1.inačica

$$\frac{3^{x}-3^{-x}}{3^{x}+3^{-x}}=\frac{4}{5} \Rightarrow \left[\begin{array}{l}\text{razlomak na lijevoj}\\ \text{strani proširimo sa } 3^{x}\end{array}\right] \Rightarrow \frac{\left(3^{x}-3^{-x}\right)\cdot 3^{x}}{\left(3^{x}+3^{-x}\right)\cdot 3^{x}}=\frac{4}{5} \Rightarrow \frac{3^{2\cdot x}-3^{0}}{3^{2\cdot x}+3^{0}}=\frac{4}{5} \Rightarrow \frac{3^{2\cdot x}-1}{3^{2\cdot x}+1}=\frac{4}{5} \Rightarrow$$

$$\Rightarrow 5\cdot\left(3^{2\cdot x}-1\right)=4\cdot\left(3^{2\cdot x}+1\right) \Rightarrow 5\cdot 3^{2\cdot x}-5=4\cdot 3^{2\cdot x}+4 \Rightarrow 5\cdot 3^{2\cdot x}-4\cdot 3^{2\cdot x}=4+5 \Rightarrow 3^{2\cdot x}=9 \Rightarrow$$

$$\Rightarrow 3^{2\cdot x}=3^{2} \Rightarrow 2\cdot x=2 \Rightarrow 2\cdot x=2\ /:2 \Rightarrow x=1.$$

2.inačica

$$\frac{3^{x}-3^{-x}}{3^{x}+3^{-x}}=\frac{4}{5} \Rightarrow \frac{3^{x}-\frac{1}{3^{x}}}{3^{x}+\frac{1}{3^{x}}}=\frac{4}{5} \Rightarrow \frac{\frac{3^{2\cdot x}-1}{3^{x}}}{\frac{3^{2\cdot x}+1}{3^{x}}}=\frac{4}{5} \Rightarrow \frac{\frac{3^{2\cdot x}-1}{\cancel{3^{x}}}}{\frac{3^{2\cdot x}+1}{\cancel{3^{x}}}}=\frac{4}{5} \Rightarrow \frac{3^{2\cdot x}-1}{3^{2\cdot x}+1}=\frac{4}{5} \Rightarrow$$

$$\Rightarrow 5\cdot\left(3^{2\cdot x}-1\right)=4\cdot\left(3^{2\cdot x}+1\right) \Rightarrow 5\cdot 3^{2\cdot x}-5=4\cdot 3^{2\cdot x}+4 \Rightarrow 5\cdot 3^{2\cdot x}-4\cdot 3^{2\cdot x}=4+5 \Rightarrow 3^{2\cdot x}=9 \Rightarrow$$

$$\Rightarrow 3^{2\cdot x}=3^{2} \Rightarrow 2\cdot x=2 \Rightarrow 2\cdot x=2\ /:2 \Rightarrow x=1.$$

Vježba 204

Riješi jednadžbu: $\frac{3^{x}+3^{-x}}{3^{x}-3^{-x}}=\frac{5}{4}.$

Rezultat: 1.

Zadatak 205 (Željka, srednja škola)

Riješi jednadžbu: $\frac{1}{\log_{x}2}\cdot\frac{1}{\log_{2x}2}=\frac{1}{\log_{4x}2}.$

Rješenje 205

Ponovimo!

$$\log_b a = \frac{1}{\log_a b} \quad , \quad \log_b (x \cdot y) = \log_b x + \log_b y \quad , \quad \log_b a^n = n \cdot \log_b a \quad , \quad \log_b b = 1.$$

Definicija:

$$\log_b a = c \; , \; b > 0 \; , \; b \neq 1 \; , \; a > 0 \; \Leftrightarrow \; b^c = a.$$

Logaritam broja a po bazi b je broj c kojim treba potencirati bazu b da se dobije broj a.

Mnemotehničko pravilo za pamćenje osnovne veze eksponencijalne i logaritamske funkcije:

$$\log_b a = c \qquad \underset{\rightarrow}{\log_b a = b^c} \qquad a = b^c$$

Prvo moramo napraviti diskusiju rješenja zadatka. Budući da baza logaritma mora biti pozitivan broj različit od jedan, postavit ćemo sljedeći uvjet koji traženo rješenje mora zadovoljiti:

$$\left.\begin{array}{l} x > 0 \\ x \neq 1 \\ 2 \cdot x \neq 1 \\ 4 \cdot x \neq 1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x > 0 \\ x \neq 1 \\ 2 \cdot x \neq 1 \; / : 2 \\ 4 \cdot x \neq 1 \; / : 4 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x > 0 \\ x \neq 1 \\ x \neq \frac{1}{2} \\ x \neq \frac{1}{4} \end{array}\right\}.$$

$$\frac{1}{\log_x 2} \cdot \frac{1}{\log_{2x} 2} = \frac{1}{\log_{4x} 2} \Rightarrow$$

$$\Rightarrow \log_2 x \cdot \log_2 (2 \cdot x) = \log_2 (4 \cdot x) \Rightarrow \log_2 x \cdot \left(\log_2 2 + \log_2 x\right) = \log_2 4 + \log_2 x \Rightarrow$$

$$\Rightarrow \log_2 x \cdot \left(1 + \log_2 x\right) = \log_2 2^2 + \log_2 x \Rightarrow \log_2 x + \left(\log_2 x\right)^2 = 2 \cdot \log_2 2 + \log_2 x \Rightarrow$$

$$\Rightarrow \log_2 x + \left(\log_2 x\right)^2 = 2 \cdot \log_2 2 + \log_2 x \Rightarrow \left(\log_2 x\right)^2 = 2 \cdot 1 \Rightarrow \left(\log_2 x\right)^2 = 2 \Rightarrow$$

$$\Rightarrow \left(\log_2 x\right)^2 = 2 \; / \sqrt{\;} \Rightarrow \log_2 x = \pm\sqrt{2} \Rightarrow \left.\begin{array}{l} \log_2 x = \sqrt{2} \\ \log_2 x = -\sqrt{2} \end{array}\right\} \Rightarrow \left.\begin{array}{l} x_1 = 2^{\sqrt{2}} \\ x_2 = 2^{-\sqrt{2}} \end{array}\right\}.$$

Vježba 205

Riješi jednadžbu: $\dfrac{\log_2 (2 \cdot x)}{\log_x 2} = \dfrac{1}{\log_{4x} 2}.$

Rezultat: $x_1 = 2^{\sqrt{2}} \; , \; x_2 = 2^{-\sqrt{2}}.$

Zadatak 206 (Tiny, gimnazija)

Riješi jednadžbu: $\log 10 + \frac{1}{3} \cdot \log\left(271 + 3^{2 \cdot x}\right) = 2.$

Rješenje 206

Ponovimo!

$$\log 10 = 1 \quad , \quad \log \sqrt[n]{a} = \frac{1}{n} \cdot \log a \quad , \quad \left(\sqrt[3]{a}\right)^3 = a \quad , \quad \log 1000 = 3 \quad , \quad \log f(x) = \log g(x) \Rightarrow f(x) = g(x).$$

$$a^{f(x)} = a^{g(x)} \Rightarrow f(x) = g(x).$$

1.inačica

$$\log 10+\frac{1}{3}\cdot\log\left(271+3^{2\cdot x}\right)=2 \Rightarrow 1+\frac{1}{3}\cdot\log\left(271+3^{2\cdot x}\right)=2 \Rightarrow \frac{1}{3}\cdot\log\left(271+3^{2\cdot x}\right)=2-1 \Rightarrow$$

$$\Rightarrow \frac{1}{3}\cdot\log\left(271+3^{2\cdot x}\right)=1 \Rightarrow \frac{1}{3}\cdot\log\left(271+3^{2\cdot x}\right)=1\ /\cdot 3 \Rightarrow \log\left(271+3^{2\cdot x}\right)=3 \Rightarrow$$

$$\Rightarrow \log\left(271+3^{2\cdot x}\right)=\log 1000 \Rightarrow 271+3^{2\cdot x}=1000 \Rightarrow 3^{2\cdot x}=1000-271 \Rightarrow 3^{2\cdot x}=729 \Rightarrow$$

$$\Rightarrow 3^{2\cdot x}=3^{6} \Rightarrow 2\cdot x=6 \Rightarrow 2\cdot x=6\ /:2 \Rightarrow x=3.$$

2.inačica

$$\log 10+\frac{1}{3}\cdot\log\left(271+3^{2\cdot x}\right)=2 \Rightarrow 1+\frac{1}{3}\cdot\log\left(271+3^{2\cdot x}\right)=2 \Rightarrow \frac{1}{3}\cdot\log\left(271+3^{2\cdot x}\right)=2-1 \Rightarrow$$

$$\Rightarrow \frac{1}{3}\cdot\log\left(271+3^{2\cdot x}\right)=1 \Rightarrow \log\sqrt[3]{271+3^{2\cdot x}}=1 \Rightarrow \log\sqrt[3]{271+3^{2\cdot x}}=\log 10 \Rightarrow \sqrt[3]{271+3^{2\cdot x}}=10 \Rightarrow$$

$$\Rightarrow \sqrt[3]{271+3^{2\cdot x}}=10\ /^{3} \Rightarrow \left(\sqrt[3]{271+3^{2\cdot x}}\right)^{3}=10^{3} \Rightarrow 271+3^{2\cdot x}=1000 \Rightarrow$$

$$\Rightarrow 3^{2\cdot x}=1000-271 \Rightarrow 3^{2\cdot x}=729 \Rightarrow 3^{2\cdot x}=3^{6} \Rightarrow 2\cdot x=6 \Rightarrow 2\cdot x=6\ /:2 \Rightarrow x=3.$$

Vježba 206

Riješi jednadžbu: $\log 10+\frac{1}{3}\cdot\log\left(271+3^{\sqrt{2}\cdot x}\right)=2.$

Rezultat: $3\cdot\sqrt{2}.$

Zadatak 207 (Tiny, gimnazija)

Riješi jednadžbu: $x^{2\cdot\log x-5}-0.01=0.$

Rješenje 207

Ponovimo!

$$\log 10=1\ ,\ \log a^{n}=n\cdot\log a\ ,\ \left(\sqrt[3]{a}\right)^{3}=a\ ,\ \log 1000=3\ ,\ \log f(x)=\log g(x) \Rightarrow f(x)=g(x).$$

$$a^{f(x)}=a^{g(x)} \Rightarrow f(x)=g(x).$$

Definicija:

$$\log_{b} a=c\ ,\ b>0\ ,\ b\neq 1\ ,\ a>0 \Leftrightarrow b^{c}=a\ ,\quad \log_{10} a=\log a.$$

Logaritam broja a po bazi b je broj c kojim treba potencirati bazu b da se dobije broj a.

Mnemotehničko pravilo za pamćenje osnovne veze eksponencijalne i logaritamske funkcije:

$$\log_{b} a=c \qquad \log_{b} a=b^{c} \qquad a=b^{c}$$

$$\rightarrow$$

$$x^{2\cdot\log x-5}-0.01=0 \Rightarrow x^{2\cdot\log x-5}=0.01 \Rightarrow x^{2\cdot\log x-5}=10^{-2} \Rightarrow \begin{bmatrix}\text{logaritmiramo}\\ \text{jednadžbu}\end{bmatrix} \Rightarrow$$

$$\Rightarrow x^{2\cdot\log x-5}=10^{-2}\ /\log \Rightarrow \log x^{2\cdot\log x-5}=\log 10^{-2} \Rightarrow (2\cdot\log x-5)\cdot\log x=-2\cdot\log 10 \Rightarrow$$

$$\Rightarrow (2\cdot\log x-5)\cdot\log x=-2\cdot 1 \Rightarrow (2\cdot\log x-5)\cdot\log x=-2 \Rightarrow (2\cdot\log x-5)\cdot\log x+2=0 \Rightarrow$$

$$\Rightarrow 2\cdot\log^{2} x-5\cdot\log x+2=0 \Rightarrow \begin{bmatrix}\text{supstitucija}\\ t=\log x\end{bmatrix} \Rightarrow 2\cdot t^{2}-5\cdot t+2=0 \Rightarrow \left.\begin{array}{c}2\cdot t^{2}-5\cdot t+2=0\\ a=2\ ,\ b=-5\ ,\ c=2\end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{matrix} a=2\ ,\ b=-5\ ,\ c=2 \\ t_{1,2}=\dfrac{-b\pm\sqrt{b^2-4\cdot a\cdot c}}{2\cdot a}\end{matrix}\right\} \Rightarrow t_{1,2}=\frac{5\pm\sqrt{25-4\cdot 2\cdot 2}}{2\cdot 2} \Rightarrow t_{1,2}=\frac{5\pm\sqrt{25-16}}{4} \Rightarrow t_{1,2}=\frac{5\pm\sqrt{9}}{4} \Rightarrow$$

$$\Rightarrow t_{1,2}=\frac{5\pm 3}{4} \Rightarrow \left.\begin{matrix} t_1=\dfrac{5+3}{4} \\ t_2=\dfrac{5-3}{4}\end{matrix}\right\} \Rightarrow \left.\begin{matrix} t_1=\dfrac{8}{4} \\ t_2=\dfrac{2}{4}\end{matrix}\right\} \Rightarrow \left.\begin{matrix} t_1=2 \\ t_2=\dfrac{1}{2}\end{matrix}\right\}.$$

Vraćamo se na supstituciju.

- $\left.\begin{matrix} t=\log x \\ t=2\end{matrix}\right\} \Rightarrow \log x=2 \Rightarrow x=10^2 \Rightarrow x_1=10^2 \Rightarrow x_1=100.$
- $\left.\begin{matrix} t=\log x \\ t=\dfrac{1}{2}\end{matrix}\right\} \Rightarrow \log x=\dfrac{1}{2} \Rightarrow x=10^{\frac{1}{2}} \Rightarrow x_2=10^{\frac{1}{2}} \Rightarrow x_2=\sqrt{10}.$

Provjeravamo rezultate.

$x^{2\cdot\log x-5}-0.01=0$ $x=100$	$x^{2\cdot\log x-5}-0.01=0$ $x=\sqrt{10}$
$100^{2\cdot\log 100-5}-0.01=0$ $100^{2\cdot 2-5}-0.01=0$ $100^{4-5}-0.01=0$ $100^{-1}-0.01=0$ $\frac{1}{100}-0.01=0$ $0.01-0.01=0$ $0=0$ x = 100 je rješenje	$\left(\sqrt{10}\right)^{2\cdot\log\sqrt{10}-5}-0.01=0$ $\left(10^{\frac{1}{2}}\right)^{2\cdot\log\sqrt{10}-5}-0.01=0$ $10^{\log\sqrt{10}-\frac{5}{2}}-0.01=0$ $10^{\frac{1}{2}\cdot\log 10-\frac{5}{2}}-0.01=0$ $10^{\frac{1}{2}\cdot 1-\frac{5}{2}}-0.01=0$ $10^{\frac{1}{2}-\frac{5}{2}}-0.01=0$ $10^{-\frac{4}{2}}-0.01=0$ $10^{-2}-0.01=0$ $0.01-0.01=0$ $0=0$ $x=\sqrt{10}$ je rješenje

Vježba 207

Riješi jednadžbu: $100\cdot x^{2\cdot\log x-5}-1=0.$

Rezultat: $x_1=100\ ,\ x_2=\sqrt{10}.$

Zadatak 208 (Josip, srednja škola)

Riješi jednadžbu: $\log_2 \sqrt{(1-x)^2} = 3.$

Rješenje 208

Ponovimo!

$$\log_b b^n = n \quad , \quad \sqrt{a^2} = |a| \quad , \quad \log_b f(x) = \log_b g(x) \Rightarrow f(x) = g(x).$$

Definicija logaritma:

$$\log_b a = c \ , \ b > 0 \ , \ b \neq 1 \ , \ a > 0 \Leftrightarrow b^c = a.$$

Logaritam broja a po bazi b je broj c kojim treba potencirati bazu b da se dobije broj a.

Mnemotehničko pravilo za pamćenje osnovne veze eksponencijalne i logaritamske funkcije:

$$\log_b a = c \qquad \log_b a = b^c \ \underset{\rightarrow}{} \qquad a = b^c$$

Apsolutna vrijednost realnog broja x definira se

$$|x| = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases} \quad \text{ili} \quad |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}.$$

Ako je broj x pozitivan broj, onda ga prepišemo: | x | = x, | 7 | = 7.
Ako je broj x negativan broj, onda ga pišemo s minusom: | x | = – x, | – 4 | = – (– 4) = 4.

$$\log_2 \sqrt{(1-x)^2} = 3 \Rightarrow \log_2 \sqrt{(1-x)^2} = \log_2 2^3 \Rightarrow \sqrt{(1-x)^2} = 2^3 \Rightarrow |1-x| = 8 \Rightarrow \left.\begin{matrix} 1-x=-8 \\ 1-x=8 \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{matrix} -x=-8-1 \\ -x=8-1 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} -x=-9 \\ -x=7 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} -x=-9 \ / \cdot (-1) \\ -x=7 \ / \cdot (-1) \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x_1=9 \\ x_2=-7 \end{matrix}\right\}.$$

Vježba 208

Riješi jednadžbu: $\log_2 \sqrt{(1-x)^2} = 1.$

Rezultat: $x_1 = -1 \ , \ x_2 = 3.$

Zadatak 209 (Doris, studentica)

Riješi jednadžbu : $x^{x^2 - 5 \cdot x + 6} = 1.$

Rješenje 209

Ponovimo!

Eksponencijalna jednadžba kod koje su baza i eksponent algebarske funkcije glasi:

$$(f(x))^{g(x)} = (f(x))^{h(x)}, \qquad (1)$$

gdje su f, g, h neke algebarske funkcije.
Postoje četiri mogućnosti.

① $f(x) = 0 \Rightarrow x_0$ je rješenje.

Tada je x_0 istodobno i rješenje jednadžbe (1) ako je $g(x_0) > 0$ i $h(x_0) > 0.$

② $f(x) = 1 \Rightarrow x_0$ je rješenje.

Tada je x_0 istodobno i rješenje jednadžbe (1) ako su funkcije g i h definirane za x_0.

③ $f(x) = -1 \Rightarrow x_0$ je rješenje.

Tada je x_0 istodobno i rješenje jednadžbe (1) ako su $g(x_0)$ i $h(x_0)$ cijeli brojevi jednake parnosti ili razlomci kojima je nazivnik neparan, a brojnik paran.

④ $g(x)=h(x) \Rightarrow x_0$ je rješenje.

Tada je x_0 istodobno i rješenje jednadžbe (1) ako za x_0 jednadžba (1) ima smisla.
Specijalno ako je zadana jednadžba

$$\left(f(x)\right)^{g(x)}=1, \qquad (2)$$

problem se svodi na rješavanje triju algebarskih jednadžbi.

① $f(x)=1 \Rightarrow x_0$ je rješenje.

Tada je x_0 istodobno i rješenje jednadžbe (2) ako je funkcija g definirana za x_0.

② $f(x)=-1 \Rightarrow x_0$ je rješenje.

Tada je x_0 istodobno i rješenje jednadžbe (2) ako je $g(x_0)$ paran broj.

③ $g(x)=0 \Rightarrow x_0$ je rješenje.

Tada je x_0 istodobno i rješenje jednadžbe (2) ako za x_0 jednadžba (2) ima smisla.
Rješavamo zadatak.

$$x^{x^2-5\cdot x+6}=1 \Rightarrow \left.\begin{array}{l} f(x)=x \\ g(x)=x^2-5\cdot x+6 \end{array}\right\}.$$

① $\left.\begin{array}{l} f(x)=1 \\ f(x)=x \end{array}\right\} \Rightarrow x=1 \Rightarrow x_1=1.$

Provjerimo da je funkcija $g(x) = x^2 - 5 \cdot x + 6$ definirana za $x_1 = 1$.

$$\left.\begin{array}{l} g(x)=x^2-5\cdot x+6 \\ x=1 \end{array}\right\} \Rightarrow g(1)=1^2-5\cdot 1+6 \Rightarrow g(1)=1-5+6 \Rightarrow g(1)=2.$$

Dakle, $x_1 = 1$ rješenje je zadane jednadžbe.

② $\left.\begin{array}{l} f(x)=-1 \\ f(x)=x \end{array}\right\} \Rightarrow x=-1 \Rightarrow x_2=-1.$

Provjerimo da je vrijednost funkcije $g(x) = x^2 - 5 \cdot x + 6$ za $x_2 = -1$ paran broj.

$$\left.\begin{array}{l} g(x)=x^2-5\cdot x+6 \\ x=-1 \end{array}\right\} \Rightarrow g(-1)=(-1)^2-5\cdot(-1)+6 \Rightarrow g(-1)=1+5+6 \Rightarrow g(-1)=12.$$

Dakle, $x_2 = -1$ rješenje je zadane jednadžbe.

③ $\left.\begin{array}{l} g(x)=0 \\ g(x)=x^2-5\cdot x+6 \end{array}\right\} \Rightarrow x^2-5\cdot x+6=0 \Rightarrow \left.\begin{array}{l} x^2-5\cdot x+6=0 \\ a=1\ ,\ b=-5\ ,\ c=6 \end{array}\right\} \Rightarrow$

$$\Rightarrow \left.\begin{array}{l} a=1\ ,\ b=-5\ ,\ c=6 \\ x_{3,4}=\dfrac{-b\pm\sqrt{b^2-4\cdot a\cdot c}}{2\cdot a} \end{array}\right\} \Rightarrow x_{3,4}=\frac{5\pm\sqrt{25-4\cdot 1\cdot 6}}{2\cdot 1} \Rightarrow x_{3,4}=\frac{5\pm\sqrt{25-24}}{2} \Rightarrow$$

$$\Rightarrow x_{3,4}=\frac{5\pm\sqrt{1}}{2} \Rightarrow x_{3,4}=\frac{5\pm 1}{2} \Rightarrow \left.\begin{array}{l} x_3=\dfrac{5+1}{2} \\ x_4=\dfrac{5-1}{2} \end{array}\right\} \Rightarrow \left.\begin{array}{l} x_3=\dfrac{6}{2} \\ x_4=\dfrac{4}{2} \end{array}\right\} \Rightarrow \left.\begin{array}{l} x_3=3 \\ x_4=2 \end{array}\right\}.$$

Provjerimo da zadana jednadžba ima smisla za $x_3 = 3$, $x_4 = 2$.

- $$\left.\begin{array}{l} x=3 \\ x^{x^2-5\cdot x+6}=1 \end{array}\right\} \Rightarrow 3^{3^2-5\cdot 3+6}=1 \Rightarrow 3^{9-15+6}=1 \Rightarrow 3^0=1 \Rightarrow 1=1.$$
- $$\left.\begin{array}{l} x=2 \\ x^{x^2-5\cdot x+6}=1 \end{array}\right\} \Rightarrow 2^{2^2-5\cdot 2+6}=1 \Rightarrow 2^{4-10+6}=1 \Rightarrow 2^0=1 \Rightarrow 1=1.$$

Dakle, $x_3 = 3$ i $x_4 = 2$ rješenja su zadane jednadžbe.
Zadana jednadžba ima rješenja:

$$\{-1, 1, 2, 3\}.$$

Vježba 209

$$\text{Riješi jednadžbu}: \left(x^2-x-1\right)^{x^2-1}=1.$$

Rezultat: $\{-1, 1, 2\}$.

Zadatak 210 (Zora, gimnazija)

$$\text{Riješi jednadžbu}: 4^x+9^x+25^x=6^x+10^x+15^x.$$

Rješenje 210

Ponovimo!

$$\left(a^n\right)^m=a^{n\cdot m} \quad , \quad a^n\cdot b^n=(a\cdot b)^n \quad , \quad (a-b)^2=a^2-2\cdot a\cdot b+b^2.$$

$$a^2+b^2=0 \Leftrightarrow a=b=0 \quad , \quad a^0=1.$$

$$4^x+9^x+25^x=6^x+10^x+15^x \Rightarrow \left(2^2\right)^x+\left(3^2\right)^x+\left(5^2\right)^x=(2\cdot 3)^x+(2\cdot 5)^x+(3\cdot 5)^x \Rightarrow$$

$$\Rightarrow \left(2^x\right)^2+\left(3^x\right)^2+\left(5^x\right)^2=2^x\cdot 3^x+2^x\cdot 5^x+3^x\cdot 5^x \Rightarrow$$

$$\Rightarrow \left(2^x\right)^2+\left(3^x\right)^2+\left(5^x\right)^2-2^x\cdot 3^x-2^x\cdot 5^x-3^x\cdot 5^x=0 \Rightarrow \begin{bmatrix}\text{jednakost} \\ \text{pomnožimo s } 2\end{bmatrix} \Rightarrow$$

$$\Rightarrow \left(2^x\right)^2+\left(3^x\right)^2+\left(5^x\right)^2-2^x\cdot 3^x-2^x\cdot 5^x-3^x\cdot 5^x=0 \,/\cdot 2 \Rightarrow$$

$$\Rightarrow 2\cdot\left(2^x\right)^2+2\cdot\left(3^x\right)^2+2\cdot\left(5^x\right)^2-2\cdot 2^x\cdot 3^x-2\cdot 2^x\cdot 5^x-2\cdot 3^x\cdot 5^x=0 \Rightarrow$$

$$\Rightarrow \left(2^x\right)^2-2\cdot 2^x\cdot 3^x+\left(3^x\right)^2+\left(2^x\right)^2-2\cdot 2^x\cdot 5^x+\left(5^x\right)^2+\left(3^x\right)^2-2\cdot 3^x\cdot 5^x+\left(5^x\right)^2=0 \Rightarrow$$

$$\Rightarrow\left(\left(2^x\right)^2-2\cdot 2^x\cdot 3^x+\left(3^x\right)^2\right)+\left(\left(2^x\right)^2-2\cdot 2^x\cdot 5^x+\left(5^x\right)^2\right)+\left(\left(3^x\right)^2-2\cdot 3^x\cdot 5^x+\left(5^x\right)^2\right)=0 \Rightarrow$$

$$\Rightarrow \left(2^x-3^x\right)^2+\left(2^x-5^x\right)^2+\left(3^x-5^x\right)^2=0.$$

Budući da je zbroj kvadrata jednak nuli ako i samo ako je svaki pribrojnik jednak nuli, slijedi:

$$\left.\begin{array}{l} 2^x-3^x=0 \\ 2^x-5^x=0 \\ 3^x-5^x=0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 2^x=3^x \\ 2^x=5^x \\ 3^x=5^x \end{array}\right\} \Rightarrow 2^x=3^x=5^x.$$

Jedino rješenje je $x = 0$ jer je $a^0 = 1$, $a \neq 0$.

Vježba 210

Riješi jednadžbu: $4^x + 9^x + 16^x = 6^x + 8^x + 12^x$.

Rezultat: x = 0.

Zadatak 211 (Jasna, gimnazija)

Ako su $a = \log 3$, $b = \log 5$ i $c = \log 2$, onda $\log_{40} 9$ iznosi:

$$A)\ \frac{2\cdot a}{b+3\cdot c} \qquad B)\ \frac{a}{2\cdot b+c} \qquad C)\ \frac{b}{a+3\cdot c} \qquad D)\ \frac{2\cdot b}{a+3\cdot c}$$

Rješenje 211

Ponovimo!

$$\log_b a = \frac{\log_c a}{\log_c b} \quad , \quad \log_{10} a = \log a \quad , \quad \log_b a = \frac{\log a}{\log b} \quad , \quad \log a^n = n\cdot\log a.$$

$$\log(a\cdot b) = \log a + \log b \quad , \quad \log\frac{a}{b} = \log a - \log b.$$

$$\log_{40} 9 = \frac{\log 9}{\log 40} = \frac{\log 3^2}{\log(5\cdot 8)} = \frac{2\cdot\log 3}{\log 5 + \log 8} = \frac{2\cdot\log 3}{\log 5 + \log 2^3} = \frac{2\cdot\log 3}{\log 5 + 3\cdot\log 2} = \begin{bmatrix} \text{supstitucije} \\ a = \log 3 \\ b = \log 5 \\ c = \log 2 \end{bmatrix} = \frac{2\cdot a}{b+3\cdot c}.$$

Odgovor je pod A.

Vježba 211

Ako su $a = \log 3$, $b = \log 5$ i $c = \log 2$, onda $\log_{40} 27$ iznosi:

$$A)\ \frac{3\cdot a}{b+3\cdot c} \qquad B)\ \frac{a}{2\cdot b+c} \qquad C)\ \frac{b}{a+3\cdot c} \qquad D)\ \frac{2\cdot b}{a+3\cdot c}$$

Rezultat: A.

Zadatak 212 (Maturantica, gimnazija)

Čemu je jednako $\log_2 \frac{4}{2^{x+1}}$?

$$A)\ -x-3 \qquad B)\ -x+1 \qquad C)\ 2+(x+1)^{-1} \qquad D)\ 2\cdot(x+1)^{-1}$$

Rješenje 212

Ponovimo!

$$\frac{a^n}{a^m} = a^{n-m} \quad , \quad \log_b a^n = n\cdot\log_b a \quad , \quad \log_b b = 1 \quad , \quad \log_b \frac{x}{y} = \log_b x - \log_b y.$$

Definicija logaritma:

$$\log_b a = c \ , \ b > 0 \ , \ b \neq 1 \ , \ a > 0 \ \Leftrightarrow \ b^c = a.$$

Logaritam broja a po bazi b je broj c kojim treba potencirati bazu b da se dobije broj a.

Mnemotehničko pravilo za pamćenje osnovne veze eksponencijalne i logaritamske funkcije:

$$\log_b a = c \qquad \log_b a = b^c \qquad a = b^c$$
$$\rightarrow$$

1.inačica

$$\log_2 \frac{4}{2^{x+1}} = \log_2 \frac{2^2}{2^{x+1}} = \log_2 2^{2-(x+1)} = \log_2 2^{2-x-1} = \log_2 2^{1-x} = (1-x)\cdot\log_2 2 =$$

$$=(1-x)\cdot 1=1-x=-x+1.$$

Odgovor je pod B.

2.inačica

$$\log_2 \frac{4}{2^{x+1}}=\log_2 4-\log_2 2^{x+1}=\log_2 2^2-\log_2 2^{x+1}=2\cdot\log_2 2-(x+1)\cdot\log_2 2=$$

$$=2\cdot 1-(x+1)\cdot 1=2-x-1=-x+1.$$

Odgovor je pod B.

Vježba 212

Čemu je jednako $\log_2 \frac{8}{2^{x+1}}$?

$A)\ -x+2 \qquad B)\ -x-2 \qquad C)\ 2+(x+1)^{-1} \qquad D)\ 2\cdot(x+1)^{-1}$

Rezultat: A.

Zadatak 213 (Maturantica, gimnazija)

Za neki realni broj x vrijedi da je $\log_3 x=2.$ Koliko je tada $\log_x 9$?

$A)\ 1 \qquad B)\ 2 \qquad C)\ 3 \qquad D)\ 4$

Rješenje 213

Ponovimo!

$$\log_b b=1 \quad , \quad \log_b a=\frac{\log_c a}{\log_c b} \quad , \quad \log_{10} a=\log a \quad , \quad \log_b a=\frac{\log a}{\log b} \quad , \quad \log a^n=n\cdot\log a.$$

$$\log f(x)=\log g(x) \;\Rightarrow\; f(x)=g(x) \quad , \quad \log_b a=\frac{1}{\log_a b} \quad , \quad \log_{b^n} a=\frac{1}{n}\cdot\log_b a.$$

Definicija logaritma:

$$\log_b a=c \;,\; b>0 \;,\; b\neq 1 \;,\; a>0 \;\Leftrightarrow\; b^c=a.$$

Logaritam broja a po bazi b je broj c kojim treba potencirati bazu b da se dobije broj a.

Mnemotehničko pravilo za pamćenje osnovne veze eksponencijalne i logaritamske funkcije:

$$\log_b a=c \qquad \log_b a=b^c \qquad a=b^c$$
$$\rightarrow$$

1.inačica

Iz

$$\log_3 x=2$$

dobije se:

$$\log_3 x=2 \;\Rightarrow\; x=3^2 \;\Rightarrow\; x=9.$$

Tada je

$$\log_x 9=\left[\, x=9 \,\right]=\log_9 9=1.$$

Odgovor je pod A.

2.inačica

Iz

$$\log_3 x=2$$

dobije se:

$$\log_3 x = 2 \Rightarrow \frac{\log x}{\log 3} = 2 \Rightarrow \frac{\log x}{\log 3} = 2 \; / \cdot \log 3 \Rightarrow \log x = 2 \cdot \log 3.$$

Tada je

$$\log_x 9 = \frac{\log 9}{\log x} = \left[\log x = 2 \cdot \log 3 \right] = \frac{\log 9}{2 \cdot \log 3} = \frac{\log 3^2}{2 \cdot \log 3} = \frac{2 \cdot \log 3}{2 \cdot \log 3} = \frac{2 \cdot \log 3}{2 \cdot \log 3} = 1.$$

Odgovor je pod A.

3.inačica

$$\log_3 x = 2 \Rightarrow \frac{1}{\log_x 3} = 2 \Rightarrow \left[\begin{array}{l} \text{razlomak} \\ \text{proširimo sa } 2 \end{array} \right] \Rightarrow \frac{2}{2 \cdot \log_x 3} = 2 \Rightarrow \frac{2}{\log_x 3^2} = 2 \Rightarrow$$

$$\Rightarrow \frac{2}{\log_x 9} = 2 \Rightarrow \frac{2}{\log_x 9} = 2 \; / \cdot \frac{\log_x 9}{2} \Rightarrow \log_x 9 = 1.$$

Odgovor je pod A.

4.inačica

$$\log_x 9 = \frac{1}{\log_9 x} = \frac{1}{\log_{3^2} x} = \frac{1}{\frac{1}{2} \cdot \log_3 x} = \frac{2}{\log_3 x} = \left[\log_3 x = 2 \right] = \frac{2}{2} = 1.$$

Odgovor je pod A.

Vježba 213

Za neki realni broj x vrijedi da je $\log_2 x = 2.$ Koliko je tada $\log_x 16$?

$A)\ 1 \qquad B)\ 2 \qquad C)\ 3 \qquad D)\ 4$

Rezultat: B.

Zadatak 214 (Maturantica, gimnazija)

Koji od ponuđenih brojeva pripada skupu rješenja nejednadžbe $0.25^{x-1} > 4^{x-3}$?

$A)\ 1.5 \qquad B)\ 2.5 \qquad C)\ 3.5 \qquad D)\ 4.5$

Rješenje 214

Ponovimo!

$$a^{-n} = \frac{1}{a^n} \quad , \quad \left(a^n\right)^m = a^{n \cdot m} \quad , \quad a^{f(x)} > a^{g(x)} \Rightarrow f(x) > g(x) \; , \; a > 1.$$

$$0.25^{x-1} > 4^{x-3} \Rightarrow \left(\frac{25}{100}\right)^{x-1} > 4^{x-3} \Rightarrow \left(\frac{1}{4}\right)^{x-1} > 4^{x-3} \Rightarrow \left(4^{-1}\right)^{x-1} > 4^{x-3} \Rightarrow$$

$$\Rightarrow 4^{-x+1} > 4^{x-3} \Rightarrow -x+1 > x-3 \Rightarrow -x-x > -3-1 \Rightarrow -2 \cdot x > -4 \Rightarrow -2 \cdot x > -4 \; / : (-2) \Rightarrow$$

$$\Rightarrow x < 2 \Rightarrow x \in \langle -\infty, 2 \rangle.$$

Budući da je

$$1.5 < 2,$$

odgovor je pod A.

Vježba 214

Koji od ponuđenih brojeva pripada skupu rješenja nejednadžbe $0.5^{x-1} > 2^{x-3}$?

$A)\ 1.5 \qquad B)\ 2.5 \qquad C)\ 3.5 \qquad D)\ 4.5$

Rezultat: A.

Zadatak 215 (Ivana, gimnazija)

Izračunaj: $5^{x}-2^{x+1}=2^{x}-5^{x-1}$.

Rješenje 215

Ponovimo!

$$a^{n}\cdot a^{m}=a^{n+m} \quad , \quad a^{-n}=\frac{1}{a^{n}} \quad , \quad \frac{a^{n}}{b^{n}}=\left(\frac{a}{b}\right)^{n} \quad , \quad a^{0}=1 \quad , \quad a^{1}=a.$$

$$a^{f(x)}=a^{g(x)} \Rightarrow f(x)=g(x).$$

Zakon distribucije množenja prema zbrajanju:

$$a\cdot(b+c)=a\cdot b+a\cdot c \quad , \quad a\cdot b+a\cdot c=a\cdot(b+c).$$

1.inačica

$$5^{x}-2^{x+1}=2^{x}-5^{x-1} \Rightarrow 5^{x}+5^{x-1}=2^{x}+2^{x+1} \Rightarrow 5^{x}+5^{x}\cdot 5^{-1}=2^{x}+2^{x}\cdot 2^{1} \Rightarrow$$

$$\Rightarrow 5^{x}\cdot\left(1+5^{-1}\right)=2^{x}\cdot\left(1+2^{1}\right) \Rightarrow 5^{x}\cdot\left(1+\frac{1}{5}\right)=2^{x}\cdot(1+2) \Rightarrow 5^{x}\cdot\frac{6}{5}=2^{x}\cdot 3 \Rightarrow$$

$$\Rightarrow 5^{x}\cdot\frac{6}{5}=2^{x}\cdot 3 \;/\cdot\frac{1}{6} \Rightarrow 5^{x}\cdot\frac{1}{5}=2^{x}\cdot\frac{1}{2} \Rightarrow 5^{x}\cdot 5^{-1}=2^{x}\cdot 2^{-1} \Rightarrow 5^{x-1}=2^{x-1} \Rightarrow$$

$$\Rightarrow 5^{x-1}=2^{x-1} \;/\cdot\frac{1}{2^{x-1}} \Rightarrow \frac{5^{x-1}}{2^{x-1}}=1 \Rightarrow \left(\frac{5}{2}\right)^{x-1}=1 \Rightarrow \left(\frac{5}{2}\right)^{x-1}=\left(\frac{5}{2}\right)^{0} \Rightarrow x-1=0 \Rightarrow x=1.$$

2.inačica

$$5^{x}-2^{x+1}=2^{x}-5^{x-1} \Rightarrow 5^{x}+5^{x-1}=2^{x}+2^{x+1} \Rightarrow 5^{x}+5^{x}\cdot 5^{-1}=2^{x}+2^{x}\cdot 2^{1} \Rightarrow$$

$$\Rightarrow 5^{x}\cdot\left(1+5^{-1}\right)=2^{x}\cdot\left(1+2^{1}\right) \Rightarrow 5^{x}\cdot\left(1+\frac{1}{5}\right)=2^{x}\cdot(1+2) \Rightarrow 5^{x}\cdot\frac{6}{5}=2^{x}\cdot 3 \Rightarrow$$

$$\Rightarrow 5^{x}\cdot\frac{6}{5}=2^{x}\cdot 3 \;/\cdot\frac{1}{6} \Rightarrow 5^{x}\cdot\frac{1}{5}=2^{x}\cdot\frac{1}{2} \Rightarrow 5^{x}\cdot\frac{1}{5}=2^{x}\cdot\frac{1}{2} \;/\cdot\frac{5}{2^{x}} \Rightarrow \frac{5^{x}}{2^{x}}=\frac{5}{2} \Rightarrow$$

$$\Rightarrow \left(\frac{5}{2}\right)^{x}=\left(\frac{5}{2}\right)^{1} \Rightarrow x=1.$$

Vježba 215

Izračunaj: $5^{x-1}-2^{x+1}=2^{x}+5^{x-1}$.

Rezultat: x = 1.

Zadatak 216 (Ivana, gimnazija)

Riješi sustav: $\begin{cases} 5^{2\cdot x-1}\cdot 3^{y+1}=135 \\ 1+\log_{2}x=\log_{2}y \end{cases}$.

Rješenje 216

Ponovimo!

$$\log_{b}b=1 \quad , \quad \log_{b}(x\cdot y)=\log_{b}x+\log_{b}y.$$

$$a^{n}\cdot a^{m}=a^{n+m} \quad , \quad a^{-n}=\frac{1}{a^{n}} \quad , \quad a^{1}=a \quad , \quad a^{n}\cdot b^{n}=(a\cdot b)^{n}.$$

$$\log_b f(x)=\log_b g(x)\ \Rightarrow\ f(x)=g(x)\quad,\quad a^{f(x)}=a^{g(x)}\ \Rightarrow\ f(x)=g(x).$$

1.inačica

Transformiramo drugu jednadžbu sustava rabeći pravila za logaritme.

$$1+\log_2 x=\log_2 y\ \Rightarrow\ \log_2 2+\log_2 x=\log_2 y\ \Rightarrow\ \log_2(2\cdot x)=\log_2 y\ \Rightarrow\ 2\cdot x=y\ \Rightarrow\ y=2\cdot x.$$

Sada rješavamo sustav jednadžbi.

$$\left.\begin{array}{l}5^{2\cdot x-1}\cdot 3^{y+1}=135\\ y=2\cdot x\end{array}\right\}\Rightarrow\begin{bmatrix}\text{metoda}\\ \text{supstitucije}\end{bmatrix}\Rightarrow 5^{2\cdot x-1}\cdot 3^{2\cdot x+1}=135\ \Rightarrow\ 5^{2\cdot x}\cdot 5^{-1}\cdot 3^{2\cdot x}\cdot 3^{1}=135\ \Rightarrow$$

$$\Rightarrow\ 5^{2\cdot x}\cdot\frac{1}{5}\cdot 3^{2\cdot x}\cdot 3=135\ \Rightarrow\ 5^{2\cdot x}\cdot 3^{2\cdot x}\cdot\frac{3}{5}=135\ \Rightarrow\ (5\cdot 3)^{2\cdot x}\cdot\frac{3}{5}=135\ \Rightarrow\ 15^{2\cdot x}\cdot\frac{3}{5}=135\ \Rightarrow$$

$$\Rightarrow\ 15^{2\cdot x}\cdot\frac{3}{5}=135\ /\cdot\frac{5}{3}\ \Rightarrow\ 15^{2\cdot x}=225\ \Rightarrow\ 15^{2\cdot x}=15^{2}\ \Rightarrow\ 2\cdot x=2\ \Rightarrow\ 2\cdot x=2\ /:2\ \Rightarrow\ x=1.$$

Računamo y.

$$\left.\begin{array}{l}x=1\\ y=2\cdot x\end{array}\right\}\Rightarrow\ y=2\cdot 1\ \Rightarrow\ y=2.$$

Rješenje sustava je uređeni par:

$$(x,\ y)=(1,\ 2).$$

2.inačica

Transformiramo drugu jednadžbu sustava rabeći pravila za logaritme.

$$1+\log_2 x=\log_2 y\ \Rightarrow\ \log_2 2+\log_2 x=\log_2 y\ \Rightarrow\ \log_2(2\cdot x)=\log_2 y\ \Rightarrow\ 2\cdot x=y\ \Rightarrow\ y=2\cdot x.$$

Sada rješavamo sustav jednadžbi.

$$\left.\begin{array}{l}5^{2\cdot x-1}\cdot 3^{y+1}=135\\ y=2\cdot x\end{array}\right\}\Rightarrow\begin{bmatrix}\text{metoda}\\ \text{supstitucije}\end{bmatrix}\Rightarrow 5^{2\cdot x-1}\cdot 3^{2\cdot x+1}=135\ \Rightarrow\ 5^{2\cdot x+1-2}\cdot 3^{2\cdot x+1}=135\ \Rightarrow$$

$$\Rightarrow\ 5^{2\cdot x+1}\cdot 5^{-2}\cdot 3^{2\cdot x+1}=135\ \Rightarrow\ 5^{2\cdot x+1}\cdot 3^{2\cdot x+1}\cdot\frac{1}{5^{2}}=135\ \Rightarrow\ 5^{2\cdot x+1}\cdot 3^{2\cdot x+1}\cdot\frac{1}{25}=135\ \Rightarrow$$

$$\Rightarrow\ 5^{2\cdot x+1}\cdot 3^{2\cdot x+1}=135\ /\cdot 25\ \Rightarrow\ 5^{2\cdot x+1}\cdot 3^{2\cdot x+1}=3375\ \Rightarrow\ (5\cdot 3)^{2\cdot x+1}=15^{3}\ \Rightarrow$$

$$\Rightarrow\ 15^{2\cdot x+1}=15^{3}\ \Rightarrow\ 2\cdot x+1=3\ \Rightarrow\ 2\cdot x=3-1\ \Rightarrow\ 2\cdot x=2\ \Rightarrow\ 2\cdot x=2\ /:2\ \Rightarrow\ x=1.$$

Računamo y.

$$\left.\begin{array}{l}x=1\\ y=2\cdot x\end{array}\right\}\Rightarrow\ y=2\cdot 1\ \Rightarrow\ y=2.$$

Rješenje sustava je uređeni par:

$$(x,\ y)=(1,\ 2).$$

3.inačica

Transformiramo drugu jednadžbu sustava rabeći pravila za logaritme.

$$1+\log_2 x=\log_2 y\ \Rightarrow\ \log_2 2+\log_2 x=\log_2 y\ \Rightarrow\ \log_2(2\cdot x)=\log_2 y\ \Rightarrow\ 2\cdot x=y\ \Rightarrow\ y=2\cdot x.$$

Sada rješavamo sustav jednadžbi.

$$\left.\begin{array}{l}5^{2\cdot x-1}\cdot 3^{y+1}=135\\ y=2\cdot x\end{array}\right\}\Rightarrow\begin{bmatrix}\text{metoda}\\ \text{supstitucije}\end{bmatrix}\Rightarrow 5^{2\cdot x-1}\cdot 3^{2\cdot x+1}=135\ \Rightarrow\ 5^{2\cdot x-1}\cdot 3^{2\cdot x-1+2}=135\ \Rightarrow$$

$$\Rightarrow\ 5^{2\cdot x-1}\cdot 3^{2\cdot x-1}\cdot 3^{2}=135\ \Rightarrow\ 5^{2\cdot x-1}\cdot 3^{2\cdot x-1}\cdot 9=135\ \Rightarrow\ 5^{2\cdot x-1}\cdot 3^{2\cdot x-1}\cdot 9=135\ /:9\ \Rightarrow$$

$$\Rightarrow 5^{2\cdot x-1}\cdot 3^{2\cdot x-1}=15 \Rightarrow (5\cdot 3)^{2\cdot x-1}=15 \Rightarrow 15^{2\cdot x-1}=15^{1} \Rightarrow 2\cdot x-1=1 \Rightarrow$$

$$\Rightarrow 2\cdot x=1+1 \Rightarrow 2\cdot x=2 \Rightarrow 2\cdot x=2 \;/:2 \Rightarrow x=1.$$

Računamo y.

$$\left.\begin{array}{l} x=1 \\ y=2\cdot x \end{array}\right\} \Rightarrow y=2\cdot 1 \Rightarrow y=2.$$

Rješenje sustava je uređeni par:

$$(x,\ y)=(1,\ 2).$$

Vježba 216

Riješi sustav: $\left\{\begin{array}{l} 5^{2\cdot x-1}\cdot 3^{y+1}=135 \\ \log_2 x-\log_2 y=-1 \end{array}\right.$.

Rezultat: $(x,\ y)=(1,\ 2)$.

Zadatak 217 (David, gimnazija)

Riješi nejednadžbu: $0.8^{x}\ge\left(\frac{2}{\sqrt{5}}\right)^{3\cdot x}$.

Rješenje 217

Ponovimo!

$$\left(\sqrt{a}\right)^{2}=a \quad , \quad \left(a^{n}\right)^{m}=a^{n\cdot m} \quad , \quad \frac{a^{n}}{b^{n}}=\left(\frac{a}{b}\right)^{n}.$$

$$a^{f(x)}\ge a^{g(x)} \Rightarrow f(x)\le g(x) \;,\; 0<a<1.$$

$$0.8^{x}\ge\left(\frac{2}{\sqrt{5}}\right)^{3\cdot x} \Rightarrow \left(\frac{8}{10}\right)^{x}\ge\left(\frac{2}{\sqrt{5}}\right)^{3\cdot x} \Rightarrow \left(\frac{4}{5}\right)^{x}\ge\left(\frac{2}{\sqrt{5}}\right)^{3\cdot x} \Rightarrow \left(\left(\frac{2}{\sqrt{5}}\right)^{2}\right)^{x}\ge\left(\frac{2}{\sqrt{5}}\right)^{3\cdot x} \Rightarrow$$

$$\Rightarrow \left(\frac{2}{\sqrt{5}}\right)^{2\cdot x}\ge\left(\frac{2}{\sqrt{5}}\right)^{3\cdot x} \Rightarrow \left[\frac{2}{\sqrt{5}}<1\right] \Rightarrow 2\cdot x\le 3\cdot x \Rightarrow 2\cdot x-3\cdot x\le 0 \Rightarrow -x\le 0 \Rightarrow$$

$$\Rightarrow -x\le 0 \;/\cdot(-1) \Rightarrow x\ge 0 \Rightarrow x\in\left[\,0,\ +\infty\right\rangle.$$

Vježba 217

Riješi nejednadžbu: $0.8^{x}\le\left(\frac{2}{\sqrt{5}}\right)^{3\cdot x}$.

Rezultat: $x\in\left\langle -\infty,\ 0\,\right]$.

Zadatak 218 (Petra, gimnazija)

Riješi jednadžbu: $x^{\sqrt{x}}=\left(\sqrt{x}\right)^{x}$.

Rješenje 218

Ponovimo!

Logaritam broja a po bazi b je broj c kojim treba potencirati bazu b da se dobije broj a.

Mnemotehničko pravilo za pamćenje osnovne veze eksponencijalne i logaritamske funkcije:

$$\log_b a = c \qquad \log_b a = b^c \qquad a = b^c$$
$$\rightarrow$$

$$\log a^n = n \cdot \log a \quad , \quad \log \sqrt{a} = \frac{1}{2} \cdot \log a \quad , \quad a^0 = 1 \quad , \quad \sqrt{a^2} = a \ , \ a \ge 0.$$

$$\log_{10} x = \log x \quad , \quad a \cdot b = 0 \;\Leftrightarrow\; a = 0 \text{ ili } b = 0 \text{ ili } a = b = 0.$$

Zakon distribucije množenja prema zbrajanju:

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad , \quad a \cdot b + a \cdot c = a \cdot (b + c).$$

Uočimo da x = 0 nije rješenje jednadžbe jer se dobije neodređeni izraz 0^0.

$$x^{\sqrt{x}} = \left(\sqrt{x}\right)^x \Rightarrow \begin{bmatrix} \text{logaritmiramo} \\ \text{jednadžbu} \end{bmatrix} \Rightarrow x^{\sqrt{x}} = \left(\sqrt{x}\right)^x \ / \log \Rightarrow$$

$$\Rightarrow \log x^{\sqrt{x}} = \log\left(\sqrt{x}\right)^x \Rightarrow \sqrt{x} \cdot \log x = x \cdot \log \sqrt{x} \Rightarrow \sqrt{x} \cdot \log x = x \cdot \frac{1}{2} \cdot \log x \Rightarrow$$

$$\Rightarrow \sqrt{x} \cdot \log x - x \cdot \frac{1}{2} \cdot \log x = 0 \Rightarrow \log x \cdot \left(\sqrt{x} - \frac{1}{2} \cdot x\right) = 0 \Rightarrow \left.\begin{matrix} \log x = 0 \\ \sqrt{x} - \frac{1}{2} \cdot x = 0 \end{matrix}\right\}.$$

Rješavamo jednadžbu

$$\log x = 0.$$

$$\log x = 0 \Rightarrow x = 10^0 \Rightarrow x_1 = 1.$$

Rješavamo jednadžbu

$$\sqrt{x} - \frac{1}{2} \cdot x = 0.$$

Uvedemo supstituciju (zamjenu) $x = t^2$.

$$\sqrt{x} - \frac{1}{2} \cdot x = 0 \Rightarrow \begin{bmatrix} \text{supstitucija} \\ x = t^2 \end{bmatrix} \Rightarrow \sqrt{t^2} - \frac{1}{2} \cdot t^2 = 0 \Rightarrow t - \frac{1}{2} \cdot t^2 = 0 \Rightarrow t \cdot \left(1 - \frac{1}{2} \cdot t\right) = 0 \Rightarrow$$

$$\Rightarrow \left.\begin{matrix} t = 0 \\ 1 - \frac{1}{2} \cdot t = 0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} t_1 = 0 \\ -\frac{1}{2} \cdot t = -1 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} t_1 = 0 \\ -\frac{1}{2} \cdot t = -1 \ / \cdot (-2) \end{matrix}\right\} \Rightarrow \left.\begin{matrix} t_1 = 0 \\ t_2 = 2 \end{matrix}\right\}.$$

Vraćamo se supstituciji (zamjeni).

- $\left.\begin{matrix} x = t^2 \\ t = 0 \end{matrix}\right\} \Rightarrow x = 0^2 \Rightarrow x_1 = 0$ nema smisla.
- $\left.\begin{matrix} x = t^2 \\ t = 2 \end{matrix}\right\} \Rightarrow x = 2^2 \Rightarrow x_2 = 4.$

Rješenja jednadžbe su: $x_1 = 1$, $x_2 = 4$.

Vježba 218

Riješi jednadžbu: $\left(\sqrt{x}\right)^x - x^{\sqrt{x}} = 0.$

Rezultat: $x_1 = 1$, $x_2 = 4$.

Zadatak 219 (Petra, gimnazija)

Riješi jednadžbu: $\sqrt{\log x} = \log \sqrt{x}.$

Rješenje 219

Ponovimo!

Logaritam broja a po bazi b je broj c kojim treba potencirati bazu b da se dobije broj a.

Mnemotehničko pravilo za pamćenje osnovne veze eksponencijalne i logaritamske funkcije:

$$\log_b a = c \qquad \log_b a = b^c \; \to \qquad a = b^c$$

$$\log a^n = n \cdot \log a \quad , \quad \log \sqrt{a} = \frac{1}{2} \cdot \log a \quad , \quad a^0 = 1 \quad , \quad \sqrt{a^2} = a \; , \; a \ge 0.$$

$$\log 100 = 2 \quad , \quad \log 10\,000 = 4 \quad , \quad \log_{10} x = \log x \quad , \quad a \cdot b = 0 \;\Leftrightarrow\; a = 0 \text{ ili } b = 0 \text{ ili } a = b = 0.$$

Zakon distribucije množenja prema zbrajanju:

$$a \cdot (b + c) = a \cdot b + a \cdot c \quad , \quad a \cdot b + a \cdot c = a \cdot (b + c).$$

$$\sqrt{\log x} = \log \sqrt{x} \;\Rightarrow\; \sqrt{\log x} = \frac{1}{2} \cdot \log x.$$

Uvedemo supstituciju (zamjenu) $\log x = t^2$.

$$\sqrt{\log x} = \frac{1}{2} \cdot \log x \;\Rightarrow\; \begin{bmatrix} \text{supstitucija} \\ \log x = t^2 \end{bmatrix} \;\Rightarrow\; \sqrt{t^2} = \frac{1}{2} \cdot t^2 \;\Rightarrow\; t = \frac{1}{2} \cdot t^2 \;\Rightarrow\; t - \frac{1}{2} \cdot t^2 = 0 \;\Rightarrow$$

$$\Rightarrow\; t \cdot \left(1 - \frac{1}{2} \cdot t\right) = 0 \;\Rightarrow\; \left.\begin{array}{l} t = 0 \\ 1 - \frac{1}{2} \cdot t = 0 \end{array}\right\} \;\Rightarrow\; \left.\begin{array}{l} t_1 = 0 \\ -\frac{1}{2} \cdot t = -1 \end{array}\right\} \;\Rightarrow\; \left.\begin{array}{l} t_1 = 0 \\ -\frac{1}{2} \cdot t = -1 \; / \cdot (-2) \end{array}\right\} \;\Rightarrow\; \left.\begin{array}{l} t_1 = 0 \\ t_2 = 2 \end{array}\right\}.$$

Vraćamo se supstituciji (zamjeni).

- $\left.\begin{array}{l} \log x = t^2 \\ t = 0 \end{array}\right\} \;\Rightarrow\; \log x = 0^2 \;\Rightarrow\; \log x = 0 \;\Rightarrow\; x = 10^0 \;\Rightarrow\; x_1 = 1.$
- $\left.\begin{array}{l} \log x = t^2 \\ t = 2 \end{array}\right\} \;\Rightarrow\; \log x = 2^2 \;\Rightarrow\; \log x = 4 \;\Rightarrow\; x = 10^4 \;\Rightarrow\; x_2 = 10\,000.$

Provjera!

Ovi rezultati moraju se uvrstiti u početnu jednadžbu da bi se provjerilo jesu li oni njezina rješenja.

$$\sqrt{\log x} = \log \sqrt{x}$$

$\sqrt{\log x} = \log \sqrt{x}$ $x_1 = 1$	$\sqrt{\log x} = \log \sqrt{x}$ $x_2 = 10000$
$\sqrt{\log 1} = \log \sqrt{1}$ $\sqrt{0} = \log 1$ $0 = 0$ x_1 jest rješenje	$\sqrt{\log 10\,000} = \log \sqrt{10\,000}$ $\sqrt{4} = \log 100$ $2 = 2$ x_2 jest rješenje

Rješenja jednadžbe su: $x_1 = 1$, $x_2 = 10000$.

Vježba 219

Riješi jednadžbu: $\log\sqrt{x}-\sqrt{\log x}=0.$

Rezultat: $x_1 = 1,\ x_2 = 10000.$

Zadatak 220 (Marta, gimnazija)

Riješi jednadžbu: $\log x+\log x^2+\log x^3+\log x^4+\ \dots\ +\log x^{100}=5050,\ \left(x\neq 0\right).$

Rješenje 220

Ponovimo!

$$a^n\cdot a^m=a^{n+m}\quad,\quad 1+2+3+4+5+\ldots+n=\frac{n\cdot\left(n+1\right)}{2}.$$

Logaritam broja a po bazi b je broj c kojim treba potencirati bazu b da se dobije broj a.

Mnemotehničko pravilo za pamćenje osnovne veze eksponencijalne i logaritamske funkcije:

$$\log_b a=c\qquad \log_b a=b^c\ \rightarrow\qquad a=b^c$$

$$\log\left(a\cdot b\right)=\log a+\log b\quad,\quad \log a^n=n\cdot\log a\quad,\quad a^1=a\quad,\quad \log_{10}x=\log x.$$

Zakon distribucije množenja prema zbrajanju:

$$a\cdot\left(b+c\right)=a\cdot b+a\cdot c\quad,\quad a\cdot b+a\cdot c=a\cdot\left(b+c\right).$$

1.inačica

$$\log x+\log x^2+\log x^3+\log x^4+\ \dots\ +\log x^{100}=5050\ \Rightarrow$$

$$\Rightarrow\ \log\left(x^1\cdot x^2\cdot x^3\cdot x^4\cdot\ \dots\ \cdot x^{100}\right)=5050\ \Rightarrow\ \log x^{1+2+3+4+\ldots+100}=5050\ \Rightarrow$$

$$\Rightarrow\ \left(1+2+3+4+\ldots+100\right)\cdot\log x=5050\ \Rightarrow\ \frac{100\cdot\left(100+1\right)}{2}\cdot\log x=5050\ \Rightarrow\ \frac{100\cdot 101}{2}\cdot\log x=5050\ \Rightarrow$$

$$\Rightarrow\ \frac{100\cdot 101}{2}\cdot\log x=5050\ \Rightarrow\ 50\cdot 101\cdot\log x=5050\ \Rightarrow\ 5050\cdot\log x=5050\ \Rightarrow$$

$$\Rightarrow\ 5050\cdot\log x=5050\ /:5050\ \Rightarrow\ \log x=1\ \Rightarrow\ x=10^1\ \Rightarrow\ x=10.$$

2.inačica

$$\log x+\log x^2+\log x^3+\log x^4+\ \dots\ +\log x^{100}=5050\ \Rightarrow$$

$$\Rightarrow\ \log x+2\cdot\log x+3\cdot\log x+4\cdot\log x+\ \dots\ +100\cdot\log x=5050\ \Rightarrow$$

$$\Rightarrow\ \log x\cdot\left(1+2+3+4+\ \dots\ +100\right)=5050\ \Rightarrow\ \log x\cdot\frac{100\cdot\left(100+1\right)}{2}=5050\ \Rightarrow$$

$$\Rightarrow\ \frac{100\cdot 101}{2}\cdot\log x=5050\ \Rightarrow\ \frac{100\cdot 101}{2}\cdot\log x=5050\ \Rightarrow\ 50\cdot 101\cdot\log x=5050\ \Rightarrow\ 5050\cdot\log x=5050\ \Rightarrow$$

$$\Rightarrow\ 5050\cdot\log x=5050\ /:5050\ \Rightarrow\ \log x=1\ \Rightarrow\ x=10^1\ \Rightarrow\ x=10.$$

Vježba 220

Riješi jednadžbu: $\log x+\log x^2+\log x^3+\log x^4+\ \dots\ +\log x^{10}=55,\ \left(x\neq 0\right).$

Rezultat: $x = 10.$