Zadatak 101 (Sanela, Anamarija, maturantice gimnazije)

Riješi jednadžbu: $\sin\left(x-\frac{\pi}{4}\right)\cdot\cos\left(x-\frac{\pi}{4}\right)=\frac{\sqrt{2}}{4}.$

Rješenje 101

Ponovimo!

$$\sin 2\alpha=2\cdot\sin\alpha\cdot\cos\alpha.$$

$$\sin\left(x-\frac{\pi}{4}\right)\cdot\cos\left(x-\frac{\pi}{4}\right)=\frac{\sqrt{2}}{4}\ /\cdot 2\Rightarrow 2\cdot\sin\left(x-\frac{\pi}{4}\right)\cdot\cos\left(x-\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\Rightarrow\sin 2\cdot\left(x-\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}\Rightarrow$$

$$\Rightarrow\sin\left(2\cdot x-\frac{\pi}{2}\right)=\frac{\sqrt{2}}{2}\Rightarrow\left[\begin{matrix}\text{supstitucija}\\ 2\cdot x-\frac{\pi}{2}=t\end{matrix}\right]\Rightarrow\sin t=\frac{\sqrt{2}}{2}\Rightarrow\left.\begin{matrix}t_1=\frac{\pi}{4}+k\cdot 2\cdot\pi,\ k\in Z\\ t_2=\pi-\frac{\pi}{4}+k\cdot 2\cdot\pi,\ k\in Z\end{matrix}\right\}\Rightarrow$$

$$\Rightarrow\left.\begin{matrix}t_1=\frac{\pi}{4}+k\cdot 2\cdot\pi\\ t_2=\frac{3\cdot\pi}{4}+k\cdot 2\cdot\pi\end{matrix}\right\}.$$

Vraćamo se supstituciji:

- $$\left.\begin{matrix}t=\frac{\pi}{4}+k\cdot 2\cdot\pi\\ 2\cdot x-\frac{\pi}{2}=t\end{matrix}\right\}\Rightarrow 2\cdot x-\frac{\pi}{2}=\frac{\pi}{4}+k\cdot 2\cdot\pi\Rightarrow 2\cdot x=\frac{\pi}{4}+\frac{\pi}{2}+k\cdot 2\cdot\pi\Rightarrow$$

  $$\Rightarrow 2\cdot x=\frac{\pi+2\cdot\pi}{4}+k\cdot 2\cdot\pi\Rightarrow 2\cdot x=\frac{3\cdot\pi}{4}+k\cdot 2\cdot\pi\ /:2\Rightarrow x_1=\frac{3\cdot\pi}{8}+k\cdot\pi,\ k\in Z.$$

- $$\left.\begin{matrix}t=\frac{3\cdot\pi}{4}+k\cdot 2\cdot\pi\\ 2\cdot x-\frac{\pi}{2}=t\end{matrix}\right\}\Rightarrow 2\cdot x-\frac{\pi}{2}=\frac{3\cdot\pi}{4}+k\cdot 2\cdot\pi\Rightarrow 2\cdot x=\frac{3\cdot\pi}{4}+\frac{\pi}{2}+k\cdot 2\cdot\pi\Rightarrow$$

  $$\Rightarrow 2\cdot x=\frac{3\cdot\pi+2\cdot\pi}{4}+k\cdot 2\cdot\pi\Rightarrow 2\cdot x=\frac{5\cdot\pi}{4}+k\cdot 2\cdot\pi\ /:2\Rightarrow x_2=\frac{5\cdot\pi}{8}+k\cdot\pi,\ k\in Z.$$

Vježba 101

Riješi jednadžbu: $\sin\left(x-\frac{\pi}{4}\right)\cdot\cos\left(x-\frac{\pi}{4}\right)=\frac{1}{2\cdot\sqrt{2}}.$

Rezultat: $x_1=\frac{3\cdot\pi}{8}+k\cdot\pi,\ k\in Z$, $x_2=\frac{5\cdot\pi}{8}+k\cdot\pi,\ k\in Z.$

Zadatak 102 (Igor, gimnazija)

Pojednostavnite izraz: $\sin^2\alpha+\cos\left(\frac{\pi}{3}-\alpha\right)\cdot\cos\left(\frac{\pi}{3}+\alpha\right).$

Rješenje 102

Ponovimo!

$$\cos x\cdot\cos y=\frac{1}{2}\cdot\left[\cos(x+y)+\cos(x-y)\right]\quad,\quad\cos^2 x+\sin^2 x=1\quad,\quad\cos(\pi-\alpha)=-\cos\alpha$$

$$\cos(-x)=\cos x\quad,\quad\cos 2x=\cos^2 x-\sin^2 x.$$

$$\sin^2\alpha+\cos\left(\frac{\pi}{3}-\alpha\right)\cdot\cos\left(\frac{\pi}{3}+\alpha\right)=\sin^2\alpha+\frac{1}{2}\cdot\left[\cos\left(\frac{\pi}{3}-\alpha+\frac{\pi}{3}+\alpha\right)+\cos\left(\frac{\pi}{3}-\alpha-\frac{\pi}{3}-\alpha\right)\right]=$$

$$=\sin^2\alpha+\frac{1}{2}\cdot\left[\cos\frac{2\cdot\pi}{3}+\cos(-2\cdot\alpha)\right]=\sin^2\alpha+\frac{1}{2}\cdot\left[\cos\left(\pi-\frac{\pi}{3}\right)+\cos 2\alpha\right]=$$

$$=\sin^2\alpha+\frac{1}{2}\cdot\left[-\cos\frac{\pi}{3}+\cos^2\alpha-\sin^2\alpha\right]=\sin^2\alpha+\frac{1}{2}\cdot\left[-\frac{1}{2}+1-\sin^2\alpha-\sin^2\alpha\right]=$$

$$=\sin^2\alpha+\frac{1}{2}\cdot\left[\frac{1}{2}-2\cdot\sin^2\alpha\right]=\sin^2\alpha+\frac{1}{4}-\sin^2\alpha=\frac{1}{4}.$$

Vježba 102

Pojednostavnite izraz: $4\cdot\left[\sin^2\alpha+\cos\left(\frac{\pi}{3}-\alpha\right)\cdot\cos\left(\frac{\pi}{3}+\alpha\right)\right].$

Rezultat: 1.

Zadatak 103 (Ivana, gimnazija)

Riješi jednadžbu: $\sin x-\sqrt{3}\cdot\cos x=0.$

Rješenje 103

Ponovimo!

$$\cos^2 x+\sin^2 x=1 \quad , \quad \sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}.$$

1.inačica

$$\sin x-\sqrt{3}\cdot\cos x=0 \Rightarrow \sin x=\sqrt{3}\cdot\cos x.$$

Budući da funkcije sin x i cos x imaju isti predznak, rješenja jednadžbe nalaze se u prvom i trećem kvadrantu. Kvadriranjem jednadžbe dobije se:

$$\sin x=\sqrt{3}\cdot\cos x \;/^2 \Rightarrow \sin^2 x=3\cdot\cos^2 x \Rightarrow \sin^2 x=3\cdot\left(1-\sin^2 x\right) \Rightarrow \sin^2 x=3-3\cdot\sin^2 x \Rightarrow$$

$$\Rightarrow 4\cdot\sin^2 x=3 \;/:4 \Rightarrow \sin^2 x=\frac{3}{4} \;/\sqrt{\ } \Rightarrow \sin x=\pm\sqrt{\frac{3}{4}} \Rightarrow \sin x=\pm\frac{\sqrt{3}}{2} \Rightarrow \left.\begin{array}{l}\sin x=\frac{\sqrt{3}}{2}\\ \sin x=-\frac{\sqrt{3}}{2}\end{array}\right\}.$$

Za prvi kvadrant vrijedi:

$$\sin x=\frac{\sqrt{3}}{2} \Rightarrow x_1=\frac{\pi}{3}+k\cdot 2\cdot\pi,\; k\in Z.$$

Za treći kvadrant vrijedi:

$$\sin x=-\frac{\sqrt{3}}{2} \Rightarrow x_2=\frac{4\cdot\pi}{3}+k\cdot 2\cdot\pi,\; k\in Z.$$

Konačno rješenje jednadžbe glasi:

$$x=\frac{\pi}{3}+k\cdot\pi,\; k\in Z.$$

2.inačica

$$\sin x-\sqrt{3}\cdot\cos x=0 \Rightarrow \sin x=\sqrt{3}\cdot\cos x \;/:\cos x \Rightarrow \frac{\sin x}{\cos x}=\sqrt{3} \Rightarrow tg\,x=\sqrt{3} \Rightarrow x=\frac{\pi}{3}+k\cdot\pi,\; k\in Z.$$

Vježba 103

Riješi jednadžbu: $\sin x-\cos x=0.$

Rezultat: $x=\frac{\pi}{4}+k\cdot\pi,\; k\in Z.$

Zadatak 104 (Sanela, gimnazija)

Ako je $\cos^2\alpha+\cos^2\beta=a$, nađite $\cos(\alpha+\beta)\cdot\cos(\alpha-\beta)$.

Rješenje 104

Ponovimo!

$$\cos(x+y)=\cos x\cdot\cos y-\sin x\cdot\sin y \quad , \quad \cos(x-y)=\cos x\cdot\cos y+\sin x\cdot\sin y \quad , \quad \cos^2 x+\sin^2 x=1$$

$$\cos x\cdot\cos y=\frac{1}{2}\cdot\left[\cos(x+y)+\cos(x-y)\right] \quad , \quad \cos 2\cdot x=2\cdot\cos^2 x-1.$$

1.inačica

$$\cos(\alpha+\beta)\cdot\cos(\alpha-\beta)=(\cos\alpha\cdot\cos\beta-\sin\alpha\cdot\sin\beta)\cdot(\cos\alpha\cdot\cos\beta+\sin\alpha\cdot\sin\beta)=[\text{razlika kvadrata}]=$$

$$=(\cos\alpha\cdot\cos\beta)^2-(\sin\alpha\cdot\sin\beta)^2=\cos^2\alpha\cdot\cos^2\beta-\sin^2\alpha\cdot\sin^2\beta=$$

$$=\cos^2\alpha\cdot\cos^2\beta-\left(1-\cos^2\alpha\right)\cdot\left(1-\cos^2\beta\right)=\cos^2\alpha\cdot\cos^2\beta-\left(1-\cos^2\alpha-\cos^2\beta+\cos^2\alpha\cdot\cos^2\beta\right)=$$

$$=\cos^2\alpha\cdot\cos^2\beta-1+\cos^2\alpha+\cos^2\beta-\cos^2\alpha\cdot\cos^2\beta=\cos^2\alpha+\cos^2\beta-1=a-1.$$

2.inačica

$$\cos(\alpha+\beta)\cdot\cos(\alpha-\beta)=\frac{1}{2}\cdot\left[\cos(\alpha+\beta+\alpha-\beta)+\cos(\alpha+\beta-\alpha+\beta)\right]=\frac{1}{2}\cdot\left[\cos 2\alpha+\cos 2\beta\right]=$$

$$=\frac{1}{2}\cdot\left[2\cdot\cos^2\alpha-1+2\cdot\cos^2\beta-1\right]=\frac{1}{2}\cdot\left[2\cdot\cos^2\alpha+2\cdot\cos^2\beta-2\right]=\cos^2\alpha+\cos^2\beta-1=a-1.$$

Vježba 104

Ako je $\cos^2\alpha+\cos^2\beta=a+1$, nađite $\cos(\alpha+\beta)\cdot\cos(\alpha-\beta)$.

Rezultat: a.

Zadatak 105 (Ivan, građevinska škola)

Nađi broj korijena jednadžbe $\sin(\pi\cdot\cos x)=0$ koji se nalaze u intervalu $\langle 0, 2\cdot\pi\rangle$.

Rješenje 105

$$\sin(\pi\cdot\cos x)=0 \Rightarrow \pi\cdot\cos x=k\cdot\pi \;/:\pi \Rightarrow \cos x=k,\ k\in Z.$$

Budući da je $|\cos x|\le 1$, moguća su sljedeća rješenja:

- $\cos x=0 \Rightarrow x_1=\frac{\pi}{2}\ ,\ x_2=\frac{3\cdot\pi}{2}$
- $\cos x=1 \Rightarrow x_3=0\ ,\ x_4=2\cdot\pi$ nisu rješenja zbog $x\in\langle 0, 2\cdot\pi\rangle$
- $\cos x=-1 \Rightarrow x_5=\pi.$

Ukupno su 3 rješenja.

Vježba 105

Nađi broj korijena jednadžbe $\sin(\pi\cdot\cos x)=0$ koji se nalaze u segmentu $[0, 2\cdot\pi]$.

Rezultat: Ukupno 5 rješenja.

Zadatak 106 (Nada, maturantica)

Neka su α i β kutovi pravokutnog trokuta (α ≠ 90°, β ≠ 90°). Ako je $tg\,\alpha=\frac{7}{24}$, koliko je sin β?

Rješenje 106

Ponovimo!

$$ctg\left(\frac{\pi}{2}-x\right)=tg\,x \quad , \quad \cos^2 x+\sin^2 x=1.$$

$$\left.\begin{array}{l}\alpha+\beta=\frac{\pi}{2}\ ,\ tg\,\alpha=\frac{7}{24}\\ ctg\,\beta=ctg\left(\frac{\pi}{2}-\alpha\right)=tg\,\alpha\end{array}\right\}\Rightarrow ctg\,\beta=\frac{7}{24}\Rightarrow\frac{\cos\beta}{\sin\beta}=\frac{7}{24}\ /^2\ \Rightarrow\frac{\cos^2\beta}{\sin^2\beta}=\frac{49}{576}\Rightarrow$$

$$\Rightarrow 49\cdot\sin^2\beta=576\cdot\cos^2\beta\Rightarrow 49\cdot\sin^2\beta=576\cdot\left(1-\sin^2\beta\right)\Rightarrow 49\cdot\sin^2\beta=576-576\cdot\sin^2\beta\Rightarrow$$

$$\Rightarrow 49\cdot\sin^2\beta+576\cdot\sin^2\beta=576\Rightarrow 625\cdot\sin^2\beta=576\Rightarrow\sin^2\beta=\frac{576}{625}\ /\sqrt{\ }\ \Rightarrow$$

$$\Rightarrow\sin\beta=\sqrt{\frac{576}{625}}\Rightarrow\sin\beta=\frac{24}{25}.$$

Vježba 106

Neka su α i β kutovi pravokutnog trokuta (α ≠ 90°, β ≠ 90°). Ako je $tg\,\alpha=\frac{7}{24}$, koliko je cos β?

Rezultat: $\cos\beta=\frac{7}{25}.$

Zadatak 107 (Marija, ekonomska škola)

Pojednostavnite izraz: $\sin\left(\frac{\pi}{6}+\alpha\right)+\sin\left(\frac{\pi}{6}-\alpha\right).$

Rješenje 107

Ponovimo!

$$\sin(x+y)=\sin x\cdot\cos y+\cos x\cdot\sin y\ ,\quad \sin(x-y)=\sin x\cdot\cos y-\cos x\cdot\sin y$$

$$\sin x+\sin y=2\cdot\sin\frac{x+y}{2}\cdot\cos\frac{x-y}{2}.$$

1.inačica

$$\sin\left(\frac{\pi}{6}+\alpha\right)+\sin\left(\frac{\pi}{6}-\alpha\right)=\sin\frac{\pi}{6}\cdot\cos\alpha+\cos\frac{\pi}{6}\cdot\sin\alpha+\sin\frac{\pi}{6}\cdot\cos\alpha-\cos\frac{\pi}{6}\cdot\sin\alpha=$$

$$=2\cdot\sin\frac{\pi}{6}\cdot\cos\alpha=2\cdot\frac{1}{2}\cdot\cos\alpha=\cos\alpha.$$

2.inačica

$$\sin\left(\frac{\pi}{6}+\alpha\right)+\sin\left(\frac{\pi}{6}-\alpha\right)=2\cdot\sin\frac{\frac{\pi}{6}+\alpha+\frac{\pi}{6}-\alpha}{2}\cdot\cos\frac{\frac{\pi}{6}+\alpha-\frac{\pi}{6}+\alpha}{2}=2\cdot\sin\frac{2\cdot\frac{\pi}{6}}{2}\cdot\cos\frac{2\cdot\alpha}{2}=$$

$$=2\cdot\sin\frac{\pi}{6}\cdot\cos\alpha=2\cdot\frac{1}{2}\cdot\cos\alpha=\cos\alpha.$$

Vježba 107

Pojednostavnite izraz: $\cos\left(\frac{\pi}{3}+\alpha\right)+\cos\left(\frac{\pi}{3}-\alpha\right).$

Rezultat: cos α.

Zadatak 108 (Rea, gimnazija)

Pojednostavnite izraz: $\frac{4\cdot\sin\alpha\cdot\cos\alpha\cdot\cos 2\alpha}{\cos^2 2\alpha-\sin^2 2\alpha}\cdot ctg\,4\alpha.$

Rješenje 108

Ponovimo!

$\sin 2x = 2\cdot\sin x\cdot\cos x \quad , \quad \cos 2x = \cos^2 x - \sin^2 x \quad , \quad tg\, x = \frac{\sin x}{\cos x} \quad , \quad tg\, x\cdot ctg\, x = 1.$

$$\frac{4\cdot\sin\alpha\cdot\cos\alpha\cdot\cos 2\alpha}{\cos^2 2\alpha - \sin^2 2\alpha}\cdot ctg\, 4\alpha = \frac{2\cdot 2\cdot\sin\alpha\cdot\cos\alpha\cdot\cos 2\alpha}{\cos 4\alpha}\cdot ctg\, 4\alpha = \frac{2\cdot\sin 2\alpha\cdot\cos 2\alpha}{\cos 4\alpha}\cdot ctg\, 4\alpha =$$

$$= \frac{\sin 4\alpha}{\cos 4\alpha}\cdot ctg\, 4\alpha = tg\, 4\alpha\cdot ctg\, 4\alpha = 1.$$

Vježba 108

Pojednostavnite izraz: $\frac{\cos^2 2\alpha - \sin^2 2\alpha}{4\cdot\sin\alpha\cdot\cos\alpha\cdot\cos 2\alpha}\cdot tg\, 4\alpha.$

Rezultat: 1.

Zadatak 109 (Rea, gimnazija)

Nađite rješenja jednadžbe: $\sin x\cdot\cos 3\cdot x = \cos x\cdot\sin 3\cdot x.$

Rješenje 109

Ponovimo!

$\sin(\alpha-\beta) = \sin\alpha\cdot\cos\beta - \cos\alpha\cdot\sin\beta \quad , \quad \sin(-\alpha) = -\sin\alpha \quad , \quad \sin\alpha\cdot\cos\beta = \frac{1}{2}\cdot\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right].$

1.inačica

$$\sin x\cdot\cos 3\cdot x = \cos x\cdot\sin 3\cdot x \Rightarrow \sin x\cdot\cos 3\cdot x - \cos x\cdot\sin 3\cdot x = 0 \Rightarrow \sin(x - 3\cdot x) = 0 \Rightarrow \sin(-2\cdot x) = 0 \Rightarrow$$

$$\Rightarrow -\sin 2\cdot x = 0 \;/\cdot(-1) \Rightarrow \sin 2\cdot x = 0 \Rightarrow 2\cdot x = k\cdot\pi \;/:2 \Rightarrow x = k\cdot\frac{\pi}{2},\, k\in Z.$$

2.inačica

$$\sin x\cdot\cos 3\cdot x = \cos x\cdot\sin 3\cdot x \Rightarrow \sin x\cdot\cos 3\cdot x = \cos x\cdot\sin 3\cdot x \;/\cdot\frac{1}{\cos x\cdot\cos 3\cdot x} \Rightarrow$$

$$\Rightarrow \frac{\sin x\cdot\cos 3\cdot x}{\cos x\cdot\cos 3\cdot x} = \frac{\cos x\cdot\sin 3\cdot x}{\cos x\cdot\cos 3\cdot x} \Rightarrow \frac{\sin x}{\cos x} = \frac{\sin 3\cdot x}{\cos 3\cdot x} \Rightarrow tg\, x = tg\, 3\cdot x \Rightarrow tg\, 3\cdot x = tg\, x \Rightarrow$$

$$\Rightarrow 3\cdot x = x + k\cdot\pi \Rightarrow 3\cdot x - x = k\cdot\pi \Rightarrow 2\cdot x = k\cdot\pi \;/:2 \Rightarrow x = k\cdot\frac{\pi}{2},\, k\in Z.$$

3.inačica

$$\sin x\cdot\cos 3\cdot x = \cos x\cdot\sin 3\cdot x \Rightarrow \frac{1}{2}\cdot\left[\sin(x+3\cdot x) + \sin(x-3\cdot x)\right] = \frac{1}{2}\cdot\left[\sin(3\cdot x + x) + \sin(3\cdot x - x)\right] \Rightarrow$$

$$\Rightarrow \frac{1}{2}\cdot\left[\sin 4\cdot x + \sin(-2\cdot x)\right] = \frac{1}{2}\cdot\left[\sin 4\cdot x + \sin 2\cdot x\right] \;/\cdot 2 \Rightarrow$$

$$\Rightarrow \sin 4\cdot x + \sin(-2\cdot x) = \sin 4\cdot x + \sin 2\cdot x \Rightarrow \sin(-2\cdot x) = \sin 2\cdot x \Rightarrow -\sin 2\cdot x = \sin 2\cdot x \Rightarrow$$

$$\Rightarrow -\sin 2\cdot x - \sin 2\cdot x = 0 \Rightarrow -2\cdot\sin 2\cdot x = 0 \;/:(-2) \Rightarrow \sin 2\cdot x = 0 \Rightarrow 2\cdot x = k\cdot\pi \;/:2 \Rightarrow x = k\cdot\frac{\pi}{2},\, k\in Z.$$

Vježba 109

Nađite rješenja jednadžbe: $\sin x\cdot\cos 2x = \cos x\cdot\sin 2x.$

Rezultat: $x = k\cdot\pi,\, k\in Z.$

Zadatak 110 (Ivan, građevinska škola)

Nađite rješenja jednadžbe: $\cos 5x + 1 = 0.$

Rješenje 110

$$\cos 5x+1=0 \Rightarrow \cos 5x=-1 \Rightarrow \cos 5x=\cos\pi \Rightarrow 5\cdot x=\pi+k\cdot 2\cdot\pi \;/:5 \Rightarrow x=\frac{\pi}{5}+k\cdot\frac{2\cdot\pi}{5} \Rightarrow$$

$$\Rightarrow x=(2\cdot k+1)\cdot\frac{\pi}{5},\; k\in Z.$$

Vježba 110

Nađite rješenja jednadžbe: $\cos 4x+1=0.$

Rezultat: $x=(2\cdot k+1)\cdot\frac{\pi}{4},\; k\in Z.$

Zadatak 111 (Josip, srednja škola)

Nađite maksimum funkcije: $f(x)=\sqrt{\sin^4 x+4\cdot\cos^2 x}+\sqrt{\cos^4 x+4\cdot\sin^2 x}.$

Rješenje 111

Ponovimo!

$$\left(a^n\right)^m=a^{n\cdot m} \quad , \quad \cos^2\alpha+\sin^2\alpha=1 \quad , \quad (a-b)^2=a^2-2\cdot a\cdot b+b^2$$

$$(a+b)^2=a^2+2\cdot a\cdot b+b^2.$$

$$f(x)=\sqrt{\sin^4 x+4\cdot\cos^2 x}+\sqrt{\cos^4 x+4\cdot\sin^2 x}=\sqrt{\left(\sin^2 x\right)^2+4\cdot\cos^2 x}+\sqrt{\left(\cos^2 x\right)^2+4\cdot\sin^2 x}=$$

$$=\sqrt{\left(1-\cos^2 x\right)^2+4\cdot\cos^2 x}+\sqrt{\left(1-\sin^2 x\right)^2+4\cdot\sin^2 x}=$$

$$=\sqrt{1-2\cdot\cos^2 x+\cos^4 x+4\cdot\cos^2 x}+\sqrt{1-2\cdot\sin^2 x+\sin^4 x+4\cdot\sin^2 x}=$$

$$=\sqrt{1+2\cdot\cos^2 x+\cos^4 x}+\sqrt{1+2\cdot\sin^2 x+\sin^4 x}=\sqrt{\left(1+\cos^2 x\right)^2}+\sqrt{\left(1+\sin^2 x\right)^2}=$$

$$=1+\cos^2 x+1+\sin^2 x=2+\cos^2 x+\sin^2 x=3.$$

Vježba 111

Nađite maksimum funkcije: $f(x)=2\cdot\sqrt{\sin^4 x+4\cdot\cos^2 x}+2\cdot\sqrt{\cos^4 x+4\cdot\sin^2 x}.$

Rezultat: 6.

Zadatak 112 (Sanela, Anamarija, maturantice gimnazije)

Dokazati identitet: $\frac{\sin x+\sin 2x+\sin 3x}{\cos x+\cos 2x+\cos 3x}=tg\,2x.$

Rješenje 112

Ponovimo!

$$\sin\alpha+\sin\beta=2\cdot\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2} \quad , \quad \cos\alpha+\cos\beta=2\cdot\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2} \quad , \quad \cos(-\alpha)=\cos\alpha.$$

$$\frac{\sin x+\sin 2x+\sin 3x}{\cos x+\cos 2x+\cos 3x}=\frac{(\sin x+\sin 3x)+\sin 2x}{(\cos x+\cos 3x)+\cos 2x}=\frac{2\cdot\sin\frac{x+3x}{2}\cdot\cos\frac{x-3x}{2}+\sin 2x}{2\cdot\cos\frac{x+3x}{2}\cdot\cos\frac{x-3x}{2}+\cos 2x}=$$

$$=\frac{2\cdot\sin\frac{4x}{2}\cdot\cos\frac{-2x}{2}+\sin 2x}{2\cdot\cos\frac{4x}{2}\cdot\cos\frac{-2x}{2}+\cos 2x}=\frac{2\cdot\sin 2x\cdot\cos(-x)+\sin 2x}{2\cdot\cos 2x\cdot\cos(-x)+\cos 2x}=\frac{2\cdot\sin 2x\cdot\cos x+\sin 2x}{2\cdot\cos 2x\cdot\cos x+\cos 2x}=$$

$$=\frac{\sin 2x\cdot(2\cdot\cos x+1)}{\cos 2x\cdot(2\cdot\cos x+1)}=\frac{\sin 2x}{\cos 2x}=tg\,2x.$$ identitet je dokazan

Vježba 112

Dokazati identitet: $\frac{\cos x+\cos 2x+\cos 3x}{\sin x+\sin 2x+\sin 3x}=ctg\,2x.$

Rezultat: Dokaz analogan.

Zadatak 113 (Sanela, Anamarija, maturantice gimnazije)

Pojednostavniti izraz: $\frac{(\sin x+\cos x)^2-1}{(1+\cos x)^2-4+3\cdot\sin^2 x}.$

Rješenje 113

Ponovimo!

$(a+b)^2=a^2+2\cdot a\cdot b+b^2$, $\cos^2\alpha+\sin^2\alpha=1$, $\sin\alpha=2\cdot\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}$, $1-\cos\alpha=2\cdot\sin^2\frac{\alpha}{2}.$

$$\frac{(\sin x+\cos x)^2-1}{(1+\cos x)^2-4+3\cdot\sin^2 x}=\frac{\sin^2 x+2\cdot\sin x\cdot\cos x+\cos^2 x-1}{1+2\cdot\cos x+\cos^2 x-4+3\cdot\sin^2 x}=\frac{\left(\sin^2 x+\cos^2 x\right)+2\cdot\sin x\cdot\cos x-1}{2\cdot\cos x+\cos^2 x-3+3\cdot\sin^2 x}=$$

$$=\frac{1+2\cdot\sin x\cdot\cos x-1}{2\cdot\cos x+\cos^2 x-3\cdot\left(1-\sin^2 x\right)}=\frac{2\cdot\sin x\cdot\cos x}{2\cdot\cos x+\cos^2 x-3\cdot\cos^2 x}=\frac{2\cdot\sin x\cdot\cos x}{2\cdot\cos x-2\cdot\cos^2 x}=\frac{2\cdot\sin x\cdot\cos x}{2\cdot\cos x\cdot(1-\cos x)}=$$

$$=\frac{\sin x}{1-\cos x}=\frac{2\cdot\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{2\cdot\sin^2\frac{x}{2}}=\frac{\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{\sin\frac{x}{2}\cdot\sin\frac{x}{2}}=\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}=ctg\frac{x}{2}.$$

Vježba 113

Pojednostavniti izraz: $\frac{(1+\cos x)^2-4+3\cdot\sin^2 x}{(\sin x+\cos x)^2-1}.$

Rezultat: $tg\frac{x}{2}.$

Zadatak 114 (Sanela, Anamarija, maturantice gimnazije)

Pojednostavniti izraz: $\frac{4\cdot\sin^2 x-\sin^2 2x}{4\cdot\cos^2 x-\sin^2 2x}.$

Rješenje 114

Ponovimo!

$a^2-b^2=(a-b)\cdot(a+b)$, $\sin 2\alpha=2\cdot\sin\alpha\cdot\cos\alpha$, $\cos^2\alpha+\sin^2\alpha=1$, $a^n\cdot a^m=a^{n+m}.$

$$\frac{4\cdot\sin^2 x-\sin^2 2x}{4\cdot\cos^2 x-\sin^2 2x}=\frac{(2\cdot\sin x-\sin 2x)\cdot(2\cdot\sin x+\sin 2x)}{(2\cdot\cos x-\sin 2x)\cdot(2\cdot\cos x+\sin 2x)}=$$

$$=\frac{(2\cdot\sin x-2\cdot\sin x\cdot\cos x)\cdot(2\cdot\sin x+2\cdot\sin x\cdot\cos x)}{(2\cdot\cos x-2\cdot\sin x\cdot\cos x)\cdot(2\cdot\cos x+2\cdot\sin x\cdot\cos x)}=\frac{2\cdot\sin x\cdot(1-\cos x)\cdot 2\cdot\sin x\cdot(1+\cos x)}{2\cdot\cos x\cdot(1-\sin x)\cdot 2\cdot\cos x\cdot(1+\sin x)}=$$

$$=\frac{\sin x\cdot(1-\cos x)\cdot\sin x\cdot(1+\cos x)}{\cos x\cdot(1-\sin x)\cdot\cos x\cdot(1+\sin x)}=\frac{\sin^2 x\cdot(1-\cos x)\cdot(1+\cos x)}{\cos^2 x\cdot(1-\sin x)\cdot(1+\sin x)}=\frac{\sin^2 x\cdot\left(1-\cos^2 x\right)}{\cos^2 x\cdot\left(1-\sin^2 x\right)}=$$

$$=\frac{\sin^2 x\cdot\sin^2 x}{\cos^2 x\cdot\cos^2 x}=\frac{\sin^4 x}{\cos^4 x}=tg^4 x.$$

Vježba 114

Pojednostavniti izraz: $\frac{4\cdot\cos^2 x-\sin^2 2x}{4\cdot\sin^2 x-\sin^2 2x}$.

Rezultat: $ctg^4 x.$

Zadatak 115 (Sanela, Anamarija, maturantice gimnazije)

Pojednostavniti izraz: $\frac{\sin^2 2x-2\cdot\sin^2 x}{\sin^2 2x-2\cdot\cos^2 x}$.

Rješenje 115

Ponovimo!

$a^2-b^2=(a-b)\cdot(a+b)$, $(a\cdot b)^n=a^n\cdot b^n$, $\sin 2\alpha=2\cdot\sin\alpha\cdot\cos\alpha$, $\cos^2\alpha+\sin^2\alpha=1.$

$$\frac{\sin^2 2x-2\cdot\sin^2 x}{\sin^2 2x-2\cdot\cos^2 x}=\frac{(\sin 2x)^2-2\cdot\sin^2 x}{(\sin 2x)^2-2\cdot\cos^2 x}=\frac{(2\cdot\sin x\cdot\cos x)^2-2\cdot\sin^2 x}{(2\cdot\sin x\cdot\cos x)^2-2\cdot\cos^2 x}=\frac{4\cdot\sin^2 x\cdot\cos^2 x-2\cdot\sin^2 x}{4\cdot\sin^2 x\cdot\cos^2 x-2\cdot\cos^2 x}=$$

$$=\frac{2\cdot\sin^2 x\cdot\left(2\cdot\cos^2 x-1\right)}{2\cdot\cos^2 x\cdot\left(2\cdot\sin^2 x-1\right)}=\frac{\sin^2 x\cdot\left(2\cdot\cos^2 x-1\right)}{\cos^2 x\cdot\left(2\cdot\sin^2 x-1\right)}=\frac{\sin^2 x\cdot\left(2\cdot\cos^2 x-\cos^2 x-\sin^2 x\right)}{\cos^2 x\cdot\left(2\cdot\sin^2 x-\cos^2 x-\sin^2 x\right)}=$$

$$=\frac{\sin^2 x\cdot\left(\cos^2 x-\sin^2 x\right)}{\cos^2 x\cdot\left(\sin^2 x-\cos^2 x\right)}=\left[\begin{array}{l}\text{iz zagrade u brojniku}\\ \text{izlučimo minus}\end{array}\right]=\frac{-\sin^2 x\cdot\left(\sin^2 x-\cos^2 x\right)}{\cos^2 x\cdot\left(\sin^2 x-\cos^2 x\right)}=-\frac{\sin^2 x}{\cos^2 x}=-tg^2 x.$$

Vježba 115

Pojednostavniti izraz: $\frac{\sin^2 2x-2\cos^2 x}{\sin^2 2x-2\sin^2 x}$.

Rezultat: $-ctg^2 x.$

Zadatak 116 (Sanela, Anamarija, maturantice gimnazije)

Pojednostavniti izraz: $ctg\,x-\sqrt{1+ctg^2 x}$.

Rješenje 116

Ponovimo!

$\cos^2\alpha+\sin^2\alpha=1$, $1-\cos\alpha=2\cdot\sin^2\frac{\alpha}{2}$, $\sin\alpha=2\cdot\sin\frac{\alpha}{2}\cdot\cos\frac{\alpha}{2}$, $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$.

$$ctg\,x-\sqrt{1+ctg^2 x}=\frac{\cos x}{\sin x}-\sqrt{1+\frac{\cos^2 x}{\sin^2 x}}=\frac{\cos x}{\sin x}-\sqrt{\frac{\sin^2 x+\cos^2 x}{\sin^2 x}}=\frac{\cos x}{\sin x}-\sqrt{\frac{1}{\sin^2 x}}=$$

$$=\frac{\cos x}{\sin x}-\frac{1}{\sin x}=\frac{\cos x-1}{\sin x}=-\frac{1-\cos x}{\sin x}=-\frac{2\cdot\sin^2\frac{x}{2}}{2\cdot\sin\frac{x}{2}\cdot\cos\frac{x}{2}}=-\frac{\sin\frac{x}{2}\cdot\sin\frac{x}{2}}{\sin\frac{x}{2}\cdot\cos\frac{x}{2}}=-\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=-tg\frac{x}{2}.$$

Vježba 116

Pojednostavniti izraz: $ctg\,x+\sqrt{1+ctg^2 x}$.

Rezultat: $ctg\frac{x}{2}.$

Zadatak 117 (Igor, maturant)

Riješi nejednadžbu: $\left| tg\,x + ctg\,x \right| < \frac{4}{\sqrt{3}}.$

Rješenje 117

Ponovimo!

$$\cos^2\alpha + \sin^2\alpha = 1 \quad , \quad \sin 2\alpha = 2\cdot\sin\alpha\cdot\cos\alpha \quad , \quad \frac{a}{b} < \frac{c}{d} \Rightarrow \frac{b}{a} > \frac{d}{c}$$

$$\sqrt{a^2} = |a| \quad , \quad \left|\frac{a}{b}\right| = \frac{|a|}{|b|} \quad , \quad 1 - \sin^2\alpha = \frac{1-\cos 2\alpha}{2}.$$

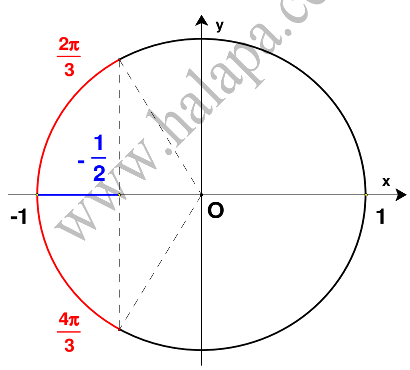
$$\left| tg\,x + ctg\,x \right| < \frac{4}{\sqrt{3}} \Rightarrow \left|\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right| < \frac{4}{\sqrt{3}} \Rightarrow \left|\frac{\sin^2 x + \cos^2 x}{\cos x\cdot\sin x}\right| < \frac{4}{\sqrt{3}} \Rightarrow \left|\frac{1}{\cos x\cdot\sin x}\right| < \frac{4}{\sqrt{3}} \Rightarrow$$

$$\Rightarrow \left|\frac{2}{2\cdot\cos x\cdot\sin x}\right| < \frac{4}{\sqrt{3}} \Rightarrow \left|\frac{2}{\sin 2x}\right| < \frac{4}{\sqrt{3}} \Rightarrow \frac{2}{|\sin 2x|} < \frac{4}{\sqrt{3}} \Rightarrow \frac{|\sin 2x|}{2} > \frac{\sqrt{3}}{4} \;/\cdot 2 \Rightarrow |\sin 2x| > \frac{\sqrt{3}}{2}.$$

Budući da je $\sqrt{a^2} = |a|$, slijedi:

$$|\sin 2x| > \frac{\sqrt{3}}{2} \Rightarrow \sqrt{\sin^2 2x} > \frac{\sqrt{3}}{2} \;/^2 \Rightarrow \sin^2 2x > \frac{3}{4} \Rightarrow \frac{1-\cos 4x}{2} > \frac{3}{4} \;/\cdot 2 \Rightarrow 1 - \cos 4x > \frac{3}{2} \Rightarrow$$

$$\Rightarrow -\cos 4x > \frac{3}{2} - 1 \Rightarrow -\cos 4x > \frac{1}{2} \;/\cdot(-1) \Rightarrow \cos 4x < -\frac{1}{2}.$$



$$\frac{2\cdot\pi}{3} + k\cdot 2\cdot\pi < 4\cdot x < \frac{4\cdot\pi}{3} + k\cdot 2\cdot\pi \;/\cdot\frac{1}{4} \Rightarrow \frac{\pi}{6} + k\cdot\frac{\pi}{2} < x < \frac{\pi}{3} + k\cdot\frac{\pi}{2} \;,\; k \in Z.$$

Vježba 117

Riješi nejednadžbu: $\left| tg\,x + ctg\,x \right| \le \frac{4}{\sqrt{3}}.$

Rezultat: $\quad \frac{\pi}{6} + k\cdot\frac{\pi}{2} \le x \le \frac{\pi}{3} + k\cdot\frac{\pi}{2} \;,\; k \in Z.$

Zadatak 118 (Elena, gimnazija)

Za kutove u trokutu vrijedi: $tg\,\alpha = \frac{2}{3}, \; \gamma = 135^0$. Nađite $tg\,\beta$.

Rješenje 118

Ponovimo!

Zbroj kutova u trokutu je 180º:

$$\alpha + \beta + \gamma = 180^\circ.$$

Tangens zbroja:

$$tg\left(x+y\right)=\frac{tg\,x+tg\,y}{1-tg\,x\cdot tg\,y}.$$

Računamo tg β:

$$\left.\begin{array}{l}\alpha+\beta+\gamma=180^{0}\\ tg\left(\alpha+\beta\right)=\frac{tg\,\alpha+tg\,\beta}{1-tg\,\alpha\cdot tg\,\beta}\end{array}\right\}\Rightarrow\left.\begin{array}{l}\alpha+\beta=180^{0}-\gamma\\ tg\left(\alpha+\beta\right)=\frac{tg\,\alpha+tg\,\beta}{1-tg\,\alpha\cdot tg\,\beta}\end{array}\right\}\Rightarrow\left.\begin{array}{l}\alpha+\beta=180^{0}-135^{0}\\ tg\left(\alpha+\beta\right)=\frac{tg\,\alpha+tg\,\beta}{1-tg\,\alpha\cdot tg\,\beta}\end{array}\right\}\Rightarrow$$

$$\Rightarrow\left.\begin{array}{l}\alpha+\beta=45^{0}\\ tg\left(\alpha+\beta\right)=\frac{tg\,\alpha+tg\,\beta}{1-tg\,\alpha\cdot tg\,\beta}\end{array}\right\}\Rightarrow tg\,45^{0}=\frac{\frac{2}{3}+tg\,\beta}{1-\frac{2}{3}\cdot tg\,\beta}\Rightarrow 1=\frac{\frac{2}{3}+tg\,\beta}{1-\frac{2}{3}\cdot tg\,\beta}\Rightarrow\frac{2}{3}+tg\,\beta=1-\frac{2}{3}\cdot tg\,\beta\Rightarrow$$

$$\Rightarrow tg\,\beta+\frac{2}{3}\cdot tg\,\beta=1-\frac{2}{3}\Rightarrow\frac{5}{3}\cdot tg\,\beta=\frac{1}{3}\,/\cdot\frac{3}{5}\Rightarrow tg\,\beta=\frac{1}{5}\Rightarrow tg\,\beta=0.2.$$

Vježba 118

Za kutove u trokutu vrijedi: $tg\,\alpha=\frac{1}{3},\ \gamma=135^{0}$. Nađite $tg\,\beta$.

Rezultat: 0.5.

Zadatak 119 (Antun, tehnička škola)

Izračunajte sin 105º – sin 15º.

Rješenje 119

Ponovimo!

$$\sin x-\sin y=2\cdot\cos\frac{x+y}{2}\cdot\sin\frac{x-y}{2}\quad,\quad\sin\left(x+y\right)=\sin x\cdot\cos y+\cos x\cdot\sin y$$

$$\sin\left(x-y\right)=\sin x\cdot\cos y-\cos x\cdot\sin y.$$

1.inačica

$$\sin105^{0}-\sin15^{0}=2\cdot\cos\frac{105^{0}+15^{0}}{2}\cdot\sin\frac{105^{0}-15^{0}}{2}=2\cdot\cos\frac{120^{0}}{2}\cdot\sin\frac{90^{0}}{2}=2\cdot\cos60^{0}\cdot\sin45^{0}=$$

$$=2\cdot\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{2}.$$

2.inačica

$$\sin105^{0}-\sin15^{0}=\sin\left(60^{0}+45^{0}\right)-\sin\left(60^{0}-45^{0}\right)=$$

$$=\sin60^{0}\cdot\cos45^{0}+\cos60^{0}\cdot\sin45^{0}-\left(\sin60^{0}\cdot\cos45^{0}-\cos60^{0}\cdot\sin45^{0}\right)=$$

$$=\sin60^{0}\cdot\cos45^{0}+\cos60^{0}\cdot\sin45^{0}-\sin60^{0}\cdot\cos45^{0}+\cos60^{0}\cdot\sin45^{0}=$$

$$=2\cdot\cos60^{0}\cdot\sin45^{0}=2\cdot\frac{1}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{2}.$$

Vježba 119

Izračunajte sin 105º – sin 75º.

Rezultat: 0.

Zadatak 120 (Antun, tehnička škola)

Koliko ima uređenih parova (x, y), $x,\ y\in\left[\,0,\ 2\pi\,\right]$ koji zadovoljavaju jednakosti

$$\cos y\cdot\cos\left(x+y\right)+\sin y\cdot\sin\left(x+y\right)=1,$$

$$\cos y\cdot\cos\left(x-y\right)-\sin y\cdot\sin\left(x-y\right)=1.$$

Rješenje 120

Ponovimo!

$\cos(\alpha+\beta)=\cos\alpha\cdot\cos\beta-\sin\alpha\cdot\sin\beta$, $\cos(\alpha-\beta)=\cos\alpha\cdot\cos\beta+\sin\alpha\cdot\sin\beta$, $\cos(-\alpha)=\cos\alpha.$

$$\left.\begin{matrix}\cos y\cdot\cos(x+y)+\sin y\cdot\sin(x+y)=1\\ \cos y\cdot\cos(x-y)-\sin y\cdot\sin(x-y)=1\end{matrix}\right\}\Rightarrow\left.\begin{matrix}\cos(y-(x+y))=1\\ \cos(y+(x-y))=1\end{matrix}\right\}\Rightarrow\left.\begin{matrix}\cos(y-x-y)=1\\ \cos(y+x-y)=1\end{matrix}\right\}\Rightarrow$$

$$\Rightarrow\left.\begin{matrix}\cos(-x)=1\\ \cos x=1\end{matrix}\right\}\Rightarrow\left.\begin{matrix}\cos x=1\\ \cos x=1\end{matrix}\right\}\Rightarrow\cos x=1\Rightarrow\left.\begin{matrix}x_1=0\\ x_2=2\cdot\pi\end{matrix}\right\}.$$

Budući da y može biti bilo koji broj iz segmenta $[0,2\pi]$, uređenih parova (x, y) ima beskonačno mnogo, ∞.

Vježba 120

Koliko ima uređenih parova (x, y), $x,\ y\in[0,\pi]$ koji zadovoljavaju jednakosti

$$\cos y\cdot\cos(x+y)+\sin y\cdot\sin(x+y)=1,$$
$$\cos y\cdot\cos(x-y)-\sin y\cdot\sin(x-y)=1.$$

Rezultat: ∞.