Zadatak 101 (Hrvoje, student VTŠ)

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to\infty}\left(\frac{x}{1+x}\right)^{x}$.

Rješenje 101

Ponovimo!

$$\frac{a-b}{n}=\frac{a}{n}-\frac{b}{n}\quad,\quad a^{n}\cdot a^{m}=a^{n+m}\quad,\quad a^{-n}=\frac{1}{a^{n}}.$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)}\ ,\ \lim_{x\to a}g(x)\neq 0\quad,\quad \lim_{x\to\infty}\frac{1}{x}=0\quad,\quad \lim_{n\to\infty}\left(1+\frac{k}{n}\right)^{n}=e^{k}.$$

$$\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^{x}=\lim_{x\to\infty}\left(\frac{1+x-1}{1+x}\right)^{x}=\lim_{x\to\infty}\left(\frac{(1+x)-1}{1+x}\right)^{x}=\lim_{x\to\infty}\left(\frac{1+x}{1+x}-\frac{1}{1+x}\right)^{x}=$$

$$=\lim_{x\to\infty}\left(\frac{1+x}{1+x}-\frac{1}{1+x}\right)^{x}=\lim_{x\to\infty}\left(1-\frac{1}{1+x}\right)^{x}=\left[\begin{array}{c}\text{supstitucija}\\ 1+x=y\ \Rightarrow\ x=y-1\\ x\to\infty\ \Rightarrow y\to\infty\end{array}\right]=\lim_{y\to\infty}\left(1-\frac{1}{y}\right)^{y-1}=$$

$$=\lim_{y\to\infty}\left(1-\frac{1}{y}\right)^{y}\cdot\left(1-\frac{1}{y}\right)^{-1}=\lim_{y\to\infty}\frac{\left(1-\frac{1}{y}\right)^{y}}{1-\frac{1}{y}}=\frac{\lim\limits_{y\to\infty}\left(1-\frac{1}{y}\right)^{y}}{\lim\limits_{y\to\infty}\left(1-\frac{1}{y}\right)}=\frac{e^{-1}}{1-0}=\frac{e^{-1}}{1}=e^{-1}=\frac{1}{e}.$$

Vježba 101

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to\infty}\left(\frac{1+x}{x}\right)^{-x}$.

Rezultat: e^{-1}.

Zadatak 102 (Hrvoje, student VTŠ)

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 0}\frac{\sin x}{1-\cos x}$.

Rješenje 102

Ponovimo!

$$\sin x=2\cdot\sin\frac{x}{2}\cdot\cos\frac{x}{2}\quad,\quad 1-\cos x=2\cdot\sin^{2}\frac{x}{2}\quad,\quad \lim_{x\to 0}ctg\ x=\infty.$$

$$\lim_{x\to 0}\frac{\sin x}{1-\cos x}=\lim_{x\to 0}\frac{2\cdot\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{2\cdot\sin^{2}\frac{x}{2}}=\lim_{x\to 0}\frac{2\cdot\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{2\cdot\sin^{2}\frac{x}{2}}=\lim_{x\to 0}\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}=\lim_{x\to 0}ctg\frac{x}{2}=\infty.$$

Vježba 102

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 0}\frac{1-\cos x}{\sin x}$.

Rezultat: 0.

Zadatak 103 (Hrvoje, student VTŠ)

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 0}\frac{\cos x-1}{x}$.

Rješenje 103

Ponovimo!

$$1-\cos x=2\cdot\sin^2\frac{x}{2}\quad,\quad \lim_{x\to 0}\frac{\sin x}{x}=1\quad,\quad \lim_{x\to 0}\sin x=0.$$

$$\lim_{x\to a} c\cdot f(x)=c\cdot\lim_{x\to a} f(x)\ ,\ c \text{ konstanta}.$$

$$\lim_{x\to 0}\frac{\cos x-1}{x}=\lim_{x\to 0}\frac{-(1-\cos x)}{x}=-\lim_{x\to 0}\frac{1-\cos x}{x}=-\lim_{x\to 0}\frac{2\cdot\sin^2\frac{x}{2}}{x}=-2\cdot\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{x}=$$

$$=-2\cdot\lim_{x\to 0}\frac{\sin\frac{x}{2}\cdot\sin\frac{x}{2}}{x}=-2\cdot\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{x}\cdot\sin\frac{x}{2}\right)=-2\cdot\lim_{x\to 0}\frac{\sin\frac{x}{2}}{x}\cdot\lim_{x\to 0}\sin\frac{x}{2}=$$

$$=-\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\lim_{x\to 0}\sin\frac{x}{2}=\left[\begin{array}{l}\text{supstitucija}\\ \frac{x}{2}=\alpha\\ x\to 0\ \Rightarrow\ \alpha\to 0\end{array}\right]=-\lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha}\cdot\lim_{x\to 0}\sin\frac{x}{2}=-1\cdot 0=0.$$

Vježba 103

Odredite graničnu vrijednost (limes): $\lim_{x\to 0}\frac{1-\cos x}{x}$.

Rezultat: 0.

Zadatak 104 (Hrvoje, student VTŠ)

Odredite graničnu vrijednost (limes): $\lim_{x\to 0}\frac{x}{1-\sqrt{1-x}}$.

Rješenje 104

Ponovimo!

Limes iracionalnog izraza često se računa i tako da prebacimo iracionalnost iz brojnika u nazivnik (da se riješimo korijena u brojniku) ili obrnuto iz nazivnika u brojnik (da se riješimo korijena u nazivniku).

Razlika kvadrata: $(a-b)\cdot(a+b)=a^2-b^2$.

1.inačica

$$\lim_{x\to 0}\frac{x}{1-\sqrt{1-x}}=\left[\begin{array}{l}\text{racionalizacija nazivnika}\\ \text{brojnik i nazivnik}\\ \text{pomnožimo sa } 1+\sqrt{1-x}\end{array}\right]=\lim_{x\to 0}\frac{x}{1-\sqrt{1-x}}\cdot\frac{1+\sqrt{1-x}}{1+\sqrt{1-x}}=\lim_{x\to 0}\frac{x\cdot\left(1+\sqrt{1-x}\right)}{1^2-\left(\sqrt{1-x}\right)^2}=$$

$$=\lim_{x\to 0}\frac{x\cdot\left(1+\sqrt{1-x}\right)}{1-(1-x)}=\lim_{x\to 0}\frac{x\cdot\left(1+\sqrt{1-x}\right)}{1-1+x}=\lim_{x\to 0}\frac{x\cdot\left(1+\sqrt{1-x}\right)}{1-1+x}=\lim_{x\to 0}\frac{x\cdot\left(1+\sqrt{1-x}\right)}{x}=$$

$$=\lim_{x\to 0}\frac{x\cdot\left(1+\sqrt{1-x}\right)}{x}=\lim_{x\to 0}\left(1+\sqrt{1-x}\right)=1+\sqrt{1-0}=1+\sqrt{1}=1+1=2.$$

2.inačica

$$\lim_{x\to 0}\frac{x}{1-\sqrt{1-x}}=\left[\begin{array}{l}\text{supstitucija}\\ t^2=1-x\ \Rightarrow\ x=1-t^2\\ x\to 0\ \Rightarrow\ t\to 1\end{array}\right]=\lim_{t\to 1}\frac{1-t^2}{1-\sqrt{t^2}}=\lim_{t\to 1}\frac{1-t^2}{1-t}=\lim_{t\to 1}\frac{(1-t)\cdot(1+t)}{1-t}=$$

$$= \lim_{t \to 1} \frac{(1-t)\cdot(1+t)}{1-t} = \lim_{t \to 1} (1+t) = 1+1 = 2.$$

Vježba 104

Odredite graničnu vrijednost (limes): $\lim_{x \to 0} \frac{1-\sqrt{1-x}}{x}.$

Rezultat: $\frac{1}{2}.$

Zadatak 105 (Hrvoje, student VTŠ)

Odredite graničnu vrijednost (limes): $\lim_{x \to 0} \frac{\sqrt{x^2+4}-2}{\sqrt{x^2+9}-3}.$

Rješenje 105

Ponovimo!

Limes iracionalnog izraza često se računa i tako da prebacimo iracionalnost iz brojnika u nazivnik (da se riješimo korijena u brojniku) ili obrnuto iz nazivnika u brojnik (da se riješimo korijena u nazivniku).

Razlika kvadrata: $(a-b)\cdot(a+b) = a^2 - b^2.$

$$\lim_{x \to 0} \frac{\sqrt{x^2+4}-2}{\sqrt{x^2+9}-3} = \begin{bmatrix} \text{racionalizacija} \\ \text{nazivnika} \end{bmatrix} = \lim_{x \to 0} \frac{\sqrt{x^2+4}-2}{\sqrt{x^2+9}-3} \cdot \frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3} =$$

$$= \lim_{x \to 0} \frac{\left(\sqrt{x^2+4}-2\right)\cdot\left(\sqrt{x^2+9}+3\right)}{x^2+9-9} = \lim_{x \to 0} \frac{\left(\sqrt{x^2+4}-2\right)\cdot\left(\sqrt{x^2+9}+3\right)}{x^2+9-9} =$$

$$= \lim_{x \to 0} \frac{\left(\sqrt{x^2+4}-2\right)\cdot\left(\sqrt{x^2+9}+3\right)}{x^2} = \lim_{x \to 0} \frac{\left(\sqrt{x^2+9}+3\right)\cdot\left(\sqrt{x^2+4}-2\right)}{x^2} = \begin{bmatrix} \text{racionalizacija} \\ \text{brojnika} \end{bmatrix} =$$

$$= \lim_{x \to 0} \frac{\left(\sqrt{x^2+9}+3\right)\cdot\left(\sqrt{x^2+4}-2\right)}{x^2} \cdot \frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2} =$$

$$= \lim_{x \to 0} \frac{\left(\sqrt{x^2+9}+3\right)\cdot\left(\left(\sqrt{x^2+4}\right)^2-2^2\right)}{x^2\cdot\left(\sqrt{x^2+4}+2\right)} = \lim_{x \to 0} \frac{\left(\sqrt{x^2+9}+3\right)\cdot\left(x^2+4-4\right)}{x^2\cdot\left(\sqrt{x^2+4}+2\right)} =$$

$$= \lim_{x \to 0} \frac{\left(\sqrt{x^2+9}+3\right)\cdot\left(x^2+4-4\right)}{x^2\cdot\left(\sqrt{x^2+4}+2\right)} == \lim_{x \to 0} \frac{\left(\sqrt{x^2+9}+3\right)\cdot x^2}{x^2\cdot\left(\sqrt{x^2+4}+2\right)} =$$

$$= \lim_{x \to 0} \frac{\left(\sqrt{x^2+9}+3\right)\cdot x^2}{x^2\cdot\left(\sqrt{x^2+4}+2\right)} = \lim_{x \to 0} \frac{\sqrt{x^2+9}+3}{\sqrt{x^2+4}+2} = \frac{\sqrt{0^2+9}+3}{\sqrt{0^2+4}+2} = \frac{\sqrt{9}+3}{\sqrt{4}+2} = \frac{3+3}{2+2} = \frac{6}{4} = \frac{3}{2}.$$

Vježba 105

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to 0} \frac{\sqrt{x^2+9}-3}{\sqrt{x^2+4}-2}.$

Rezultat: $\frac{2}{3}.$

Zadatak 106 (Hrvoje, student VTŠ)

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to \infty} \frac{x^2-3\cdot x+1}{(x-1)^3-(x+1)^3}.$

Rješenje 106

Ponovimo!

Pri traženju limesa količnika dvaju cijelih polinoma varijable x kada $x \to \infty$, preporučljivo je oba člana količnika prethodno podijeliti s x^p, gdje je p najveći eksponent potencija tih polinoma.

Također je važno zapamtiti sljedeće limese

$$\lim_{x \to \infty} \frac{1}{x} = 0 \quad , \quad \lim_{x \to \infty} \frac{1}{x^2} = 0.$$

$$(a-b)^3 = a^3 - 3\cdot a^2 \cdot b + 3\cdot a \cdot b^2 - b^3 \quad , \quad (a+b)^3 = a^3 + 3\cdot a^2 \cdot b + 3\cdot a \cdot b^2 + b^3.$$

Ako su nizovi { a_n } i { b_n } konvergentni, tada vrijede sljedeća pravila o graničnoj vrijednosti (limesu):

$$\lim_{n \to \infty} c = c \ , \ c - \text{konstanta} \quad , \quad \lim_{n \to \infty} \frac{c}{n} = 0 \ , \ c - \text{konstanta}.$$

$$\lim_{n \to \infty} (c \cdot a_n) = c \cdot \lim_{n \to \infty} a_n \ , \ c - \text{konstanta} \quad , \quad \lim_{n \to \infty} (a_n \pm b_n) = \lim_{n \to \infty} a_n \pm \lim_{n \to \infty} b_n.$$

$$\frac{a+b}{n} = \frac{a}{n} + \frac{b}{n} \quad , \quad \frac{a-b}{n} = \frac{a}{n} - \frac{b}{n} \quad , \quad \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n \to \infty} a_n}{\lim\limits_{n \to \infty} b_n} \ , \ \lim_{n \to \infty} b_n \neq 0.$$

$$\lim_{x \to \infty} \frac{x^2-3\cdot x+1}{(x-1)^3-(x+1)^3} = \lim_{x \to \infty} \frac{x^2-3\cdot x+1}{x^3-3\cdot x^2\cdot 1+3\cdot x\cdot 1^2-1^3-\left(x^3+3\cdot x^2\cdot 1+3\cdot x\cdot 1^2+1^3\right)} =$$

$$= \lim_{x \to \infty} \frac{x^2-3\cdot x+1}{x^3-3\cdot x^2\cdot 1+3\cdot x\cdot 1^2-1^3-x^3-3\cdot x^2\cdot 1-3\cdot x\cdot 1^2-1^3} =$$

$$= \lim_{x \to \infty} \frac{x^2-3\cdot x+1}{x^3-3\cdot x^2\cdot 1+3\cdot x\cdot 1^2-1^3-x^3-3\cdot x^2\cdot 1-3\cdot x\cdot 1^2-1^3} =$$

$$= \lim_{x \to \infty} \frac{x^2-3\cdot x+1}{-3\cdot x^2\cdot 1-1^3-3\cdot x^2\cdot 1-1^3} = \lim_{x \to \infty} \frac{x^2-3\cdot x+1}{-3\cdot x^2-1-3\cdot x^2-1} = \lim_{x \to \infty} \frac{x^2-3\cdot x+1}{-6\cdot x^2-2} =$$

$$= \lim_{x \to \infty} \frac{\frac{x^2-3\cdot x+1}{x^2}}{\frac{-6\cdot x^2-2}{x^2}} = \lim_{x \to \infty} \frac{\frac{x^2}{x^2}-\frac{3\cdot x}{x^2}+\frac{1}{x^2}}{\frac{-6\cdot x^2}{x^2}-\frac{2}{x^2}} = \lim_{x \to \infty} \frac{\frac{x^2}{x^2}-\frac{3\cdot x}{x^2}+\frac{1}{x^2}}{\frac{-6\cdot x^2}{x^2}-\frac{2}{x^2}} = \lim_{x \to \infty} \frac{1-\frac{3}{x}+\frac{1}{x^2}}{-6-\frac{2}{x^2}} =$$

$$= \frac{1-0+0}{-6-0} = \frac{1}{-6} = -\frac{1}{6}.$$

Vježba 106

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to\infty} \dfrac{(x-1)^3-(x+1)^3}{x^2-3\cdot x+1}$.

Rezultat: – 6.

Zadatak 107 (Hrvoje, student VTŠ)

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 1} \dfrac{x^3-3\cdot x+2}{x^4-4\cdot x+3}$.

Rješenje 107

Ponovimo!

Zakon distribucije množenja prema zbrajanju:

$$a\cdot(b+c)=a\cdot b+a\cdot c \quad , \quad a\cdot b+a\cdot c=a\cdot(b+c).$$

$$a^2-b^2=(a-b)\cdot(a+b) \quad , \quad a^3-b^3=(a-b)\cdot\left(a^2+a\cdot b+b^2\right).$$

$$\lim_{x\to a} c=c \quad , \quad c-\text{konstanta}.$$

$$\lim_{x\to a}(c\cdot f(x))=c\cdot\lim_{x\to a} f(x) \; , \; c-\text{konstanta} \quad , \quad \lim_{x\to a}(f(x)\pm g(x))=\lim_{x\to a} f(x)\pm\lim_{x\to a} g(x).$$

$$\frac{a+b}{n}=\frac{a}{n}+\frac{b}{n} \quad , \quad \frac{a-b}{n}=\frac{a}{n}-\frac{b}{n} \quad , \quad \lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} \; , \; \lim_{x\to a} g(x)\neq 0.$$

$$\lim_{x\to 1}\frac{x^3-3\cdot x+2}{x^4-4\cdot x+3}=\lim_{x\to 1}\frac{x^3-3\cdot x+2}{x^4-4\cdot x+3}=\lim_{x\to 1}\frac{x^3-x-2\cdot x+2}{x^4-x-3\cdot x+3}=\lim_{x\to 1}\frac{\left(x^3-x\right)+(-2\cdot x+2)}{\left(x^4-x\right)+(-3\cdot x+3)}=$$

$$=\lim_{x\to 1}\frac{x\cdot\left(x^2-1\right)-2\cdot(x-1)}{x\cdot\left(x^3-1\right)-3\cdot(x-1)}=\lim_{x\to 1}\frac{x\cdot(x-1)\cdot(x+1)-2\cdot(x-1)}{x\cdot(x-1)\left(x^2+x+1\right)-3\cdot(x-1)}=\begin{bmatrix}\text{u brojniku i nazivniku}\\ \text{izlučimo } x-1\end{bmatrix}=$$

$$=\lim_{x\to 1}\frac{(x-1)\cdot(x\cdot(x+1)-2)}{(x-1)\cdot\left(x\cdot\left(x^2+x+1\right)-3\right)}=\lim_{x\to 1}\frac{(x-1)\cdot(x\cdot(x+1)-2)}{(x-1)\cdot\left(x\cdot\left(x^2+x+1\right)-3\right)}=\lim_{x\to 1}\frac{x\cdot(x+1)-2}{x\cdot\left(x^2+x+1\right)-3}=$$

$$=\lim_{x\to 1}\frac{x^2+x-2}{x^3+x^2+x-3}=\lim_{x\to 1}\frac{x^2+x-2}{x^3+x^2+x-3}=\lim_{x\to 1}\frac{x^2+x-1-1}{x^3+x^2+x-1-1-1}=\lim_{x\to 1}\frac{x^2-1+x-1}{x^3-1+x^2-1+x-1}=$$

$$=\lim_{x\to 1}\frac{\left(x^2-1\right)+x-1}{\left(x^3-1\right)+\left(x^2-1\right)+x-1}=\lim_{x\to 1}\frac{(x-1)\cdot(x+1)+(x-1)}{(x-1)\cdot\left(x^2+x+1\right)+(x-1)\cdot(x+1)+(x-1)}=$$

$$=\begin{bmatrix}\text{u brojniku i nazivniku}\\ \text{izlučimo } x-1\end{bmatrix}=$$

$$=\lim_{x\to 1}\frac{(x-1)\cdot((x+1)+1)}{(x-1)\cdot\left(\left(x^2+x+1\right)+(x+1)+1\right)}=\lim_{x\to 1}\frac{(x-1)\cdot((x+1)+1)}{(x-1)\cdot\left(\left(x^2+x+1\right)+(x+1)+1\right)}=$$

$$= \lim_{x \to 1} \frac{(x+1)+1}{\left(x^2+x+1\right)+(x+1)+1} = \frac{(1+1)+1}{\left(1^2+1+1\right)+(1+1)+1} = \frac{2+1}{3+2+1} = \frac{3}{6} = \frac{1}{2}.$$

Vježba 107

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to 1} \frac{x^4-4 \cdot x+3}{x^3-3 \cdot x+2}$.

Rezultat: 2.

Zadatak 108 (Anita, gimnazija)

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to 3} \frac{x^3-3 \cdot x^2-4 \cdot x+12}{x^2-5 \cdot x+6}$.

Rješenje 108

Ponovimo!

Zakon distribucije množenja prema zbrajanju:

$$a \cdot (b+c) = a \cdot b + a \cdot c \quad , \quad a \cdot b + a \cdot c = a \cdot (b+c).$$

$$a^2-b^2 = (a-b) \cdot (a+b) \quad , \quad (a+b)^3 = a^3+3 \cdot a^2 \cdot b+3 \cdot a \cdot b^2+b^3.$$

$$(a+b)^2 = a^2+2 \cdot a \cdot b+b^2 \quad , \quad \lim_{x \to a} c = c \ , \ c - \text{konstanta}.$$

$$\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x).$$

1.inačica

$$\lim_{x \to 3} \frac{x^3-3 \cdot x^2-4 \cdot x+12}{x^2-5 \cdot x+6} = \lim_{x \to 3} \frac{x^3-3 \cdot x^2-4 \cdot x+12}{x^2-5 \cdot x+6} = \lim_{x \to 3} \frac{x^3-3 \cdot x^2-4 \cdot x+12}{x^2-3 \cdot x-2 \cdot x+6} =$$

$$= \lim_{x \to 3} \frac{x^2 \cdot (x-3)-4 \cdot (x-3)}{x \cdot (x-3)-2 \cdot (x-3)} = \lim_{x \to 3} \frac{(x-3) \cdot \left(x^2-4\right)}{(x-3) \cdot (x-2)} = \lim_{x \to 3} \frac{(x-3) \cdot \left(x^2-4\right)}{(x-3) \cdot (x-2)} = \lim_{x \to 3} \frac{x^2-4}{x-2} =$$

$$= \lim_{x \to 3} \frac{(x-2) \cdot (x+2)}{x-2} = \lim_{x \to 3} \frac{(x-2) \cdot (x+2)}{x-2} = \lim_{x \to 3} (x+2) = 3+2 = 5.$$

2.inačica

$$\lim_{x \to 3} \frac{x^3-3 \cdot x^2-4 \cdot x+12}{x^2-5 \cdot x+6} = \begin{bmatrix} \text{supstitucija} \\ x = n+3 \\ x \to 3 \ \Rightarrow \ n \to 0 \end{bmatrix} = \lim_{n \to 0} \frac{(n+3)^3-3 \cdot (n+3)^2-4 \cdot (n+3)+12}{(n+3)^2-5 \cdot (n+3)+6} =$$

$$= \lim_{n \to 0} \frac{\left(n^3+3 \cdot n^2 \cdot 3+3 \cdot n \cdot 3^2+3^3\right)-3 \cdot \left(n^2+6 \cdot n+9\right)-4 \cdot (n+3)+12}{\left(n^2+6 \cdot n+9\right)-5 \cdot (n+3)+6} =$$

$$= \lim_{n \to 0} \frac{n^3+9 \cdot n^2+27 \cdot n+27-3 \cdot n^2-18 \cdot n-27-4 \cdot n-12+12}{n^2+6 \cdot n+9-5 \cdot n-15+6} =$$

$$= \lim_{n \to 0} \frac{n^3+9 \cdot n^2+27 \cdot n+27-3 \cdot n^2-18 \cdot n-27-4 \cdot n-12+12}{n^2+6 \cdot n+9-5 \cdot n-15+6} =$$

$$= \lim_{n\to 0}\frac{n^3+9\cdot n^2+27\cdot n-3\cdot n^2-18\cdot n-4\cdot n}{n^2+6\cdot n+9-5\cdot n-15+6}=\lim_{n\to 0}\frac{n^3+6\cdot n^2+5\cdot n}{n^2+n}=\lim_{n\to 0}\frac{n\cdot\left(n^2+6\cdot n+5\right)}{n\cdot(n+1)}=$$

$$=\lim_{n\to 0}\frac{n\cdot\left(n^2+6\cdot n+5\right)}{n\cdot(n+1)}=\lim_{n\to 0}\frac{n^2+6\cdot n+5}{n+1}=\lim_{n\to 0}\frac{n^2+6\cdot n+5}{n+1}=\lim_{n\to 0}\frac{n^2+n+5\cdot n+5}{n+1}=$$

$$=\lim_{n\to 0}\frac{n\cdot(n+1)+5\cdot(n+1)}{n+1}=\lim_{n\to 0}\frac{(n+1)\cdot(n+5)}{n+1}=\lim_{n\to 0}\frac{(n+1)\cdot(n+5)}{n+1}=\lim_{n\to 0}(n+5)=0+5=5.$$

Vježba 108

Odredite graničnu vrijednost (limes): $\lim_{x\to 3}\frac{x^2-5\cdot x+6}{x^3-3\cdot x^2-4\cdot x+12}$.

Rezultat: $\frac{1}{5}$.

Zadatak 109 (Martin, gimnazija)

Odredite graničnu vrijednost (limes): $\lim_{n\to\infty}\sqrt{n}\cdot\left(\sqrt{n+1}-\sqrt{n}\right)$.

Rješenje 109

Ponovimo!

$$n=\frac{n}{1}\quad,\quad(a-b)\cdot(a+b)=a^2-b^2\quad,\quad\left(\sqrt{a}\right)^2=a\quad,\quad\frac{a+b}{n}=\frac{a}{n}+\frac{b}{n}\quad,\quad\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}.$$

Važno je zapamtiti sljedeći limes

$$\lim_{x\to\infty}\frac{1}{x}=0.$$

Limes iracionalnog izraza često se računa i tako da prebacimo iracionalnost iz brojnika u nazivnik (da se riješimo korijena u brojniku) ili obrnuto iz nazivnika u brojnik (da se riješimo korijena u nazivniku).

Ako je funkcija y = f(x) neprekidna u točki a, onda je limes funkcije u točki a jednak funkciji limesa nezavisne varijable x u točki a:

$$\lim_{x\to a}f(x)=f\left(\lim_{x\to a}x\right).$$

Ako postoje $\lim_{x\to a}f(x)$ i $\lim_{x\to a}g(x)$, tada vrijedi poučak:

$$\lim_{x\to a}\left[f(x)+g(x)\right]=\lim_{x\to a}f(x)+\lim_{x\to a}g(x).$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)}\ ,\ \lim_{x\to a}g(x)\neq 0\quad,\quad\lim_{n\to\infty}c=c\ ,\ c-\text{konstanta}.$$

$$\lim_{n\to\infty}\sqrt{n}\cdot\left(\sqrt{n+1}-\sqrt{n}\right)=\lim_{n\to\infty}\sqrt{n}\cdot\frac{\sqrt{n+1}-\sqrt{n}}{1}=\begin{bmatrix}\text{racionalizacija}\\ \text{brojnika}\end{bmatrix}=$$

$$=\lim_{n\to\infty}\sqrt{n}\cdot\frac{\sqrt{n+1}-\sqrt{n}}{1}\cdot\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\lim_{n\to\infty}\sqrt{n}\cdot\frac{\left(\sqrt{n+1}-\sqrt{n}\right)\cdot\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}}=$$

$$=\lim_{n\to\infty}\sqrt{n}\cdot\frac{\left(\sqrt{n+1}\right)^2-\left(\sqrt{n}\right)^2}{\sqrt{n+1}+\sqrt{n}}=\lim_{n\to\infty}\sqrt{n}\cdot\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}=\lim_{n\to\infty}\sqrt{n}\cdot\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}=$$

$$=\lim_{n\to\infty}\sqrt{n}\cdot\frac{1}{\sqrt{n+1}+\sqrt{n}}=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\left[\begin{array}{l}\text{brojnik i nazivnik}\\ \text{podijelimo sa }\sqrt{n}\end{array}\right]=\lim_{n\to\infty}\frac{\frac{\sqrt{n}}{\sqrt{n}}}{\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n}}}=$$

$$=\lim_{n\to\infty}\frac{\frac{\sqrt{n}}{\sqrt{n}}}{\frac{\sqrt{n+1}}{\sqrt{n}}+\frac{\sqrt{n}}{\sqrt{n}}}=\lim_{n\to\infty}\frac{\frac{\sqrt{n}}{\sqrt{n}}}{\frac{\sqrt{n+1}}{\sqrt{n}}+\frac{\sqrt{n}}{\sqrt{n}}}=\lim_{n\to\infty}\frac{1}{\frac{\sqrt{n+1}}{\sqrt{n}}+1}=\lim_{n\to\infty}\frac{1}{\sqrt{\frac{n+1}{n}}+1}=\lim_{n\to\infty}\frac{1}{\sqrt{\frac{n}{n}+\frac{1}{n}}+1}=$$

$$=\lim_{n\to\infty}\frac{1}{\sqrt{\frac{n}{n}+\frac{1}{n}}+1}=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1}=\frac{1}{\lim\limits_{n\to\infty}\sqrt{1+\frac{1}{n}}+1}=\frac{1}{\sqrt{1+0}+1}=\frac{1}{\sqrt{1}+1}=\frac{1}{1+1}=\frac{1}{2}.$$

Vježba 109

Odredite graničnu vrijednost (limes): $\lim\limits_{n\to\infty}\left(\sqrt{n^2+1}-n\right)$.

Rezultat: 0.

Zadatak 110 (Lea, gimnazija)

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to\infty}\left(\sqrt{x^2-5\cdot x+6}-x\right)$.

Rješenje 110

Ponovimo!

$$(a-b)\cdot(a+b)=a^2-b^2\quad,\quad\left(\sqrt{a}\right)^2=a\quad,\quad\frac{a+b}{n}=\frac{a}{n}+\frac{b}{n}\quad,\quad\frac{a-b}{n}=\frac{a}{n}-\frac{b}{n}.$$

$$n=\frac{n}{1}\quad,\quad\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}\quad,\quad\frac{\sqrt{a}}{b}=\sqrt{\frac{a}{b^2}}.$$

Važno je zapamtiti sljedeći limes

$$\lim_{n\to\infty}\frac{1}{n}=0.$$

Limes iracionalnog izraza često se računa i tako da prebacimo iracionalnost iz brojnika u nazivnik (da se riješimo korijena u brojniku) ili obrnuto iz nazivnika u brojnik (da se riješimo korijena u nazivniku).

Ako je funkcija y = f(x) neprekidna u točki a, onda je limes funkcije u točki a jednak funkciji limesa nezavisne varijable x u točki a:

$$\lim_{x\to a}f(x)=f\left(\lim_{x\to a}x\right).$$

Ako postoje $\lim\limits_{x\to a}f(x)$ i $\lim\limits_{x\to a}g(x)$, tada vrijedi poučak:

$$\lim_{x\to a}\left[f(x)+g(x)\right]=\lim_{x\to a}f(x)+\lim_{x\to a}g(x).$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}\ ,\ \lim_{x\to a} g(x)\neq 0 \quad , \quad \lim_{n\to\infty} c=c\ ,\ c-\text{konstanta}.$$

Ako su nizovi { a_n } i { b_n } konvergentni, tada vrijede sljedeća pravila o graničnoj vrijednosti (limesu):

$$\lim_{n\to\infty} c=c\ ,\ c-\text{konstanta} \quad , \quad \lim_{n\to\infty}\frac{c}{n}=0\ ,\ c-\text{konstanta} \quad , \quad \lim_{n\to\infty}\frac{1}{n}=0$$

$$\lim_{n\to\infty}(c\cdot a_n)=c\cdot\lim_{n\to\infty} a_n\ ,\ c-\text{konstanta} \quad , \quad \lim_{n\to\infty}(a_n\pm b_n)=\lim_{n\to\infty} a_n\pm\lim_{n\to\infty} b_n.$$

$$\lim_{n\to\infty}(a_n\cdot b_n)=\lim_{n\to\infty} a_n\cdot\lim_{n\to\infty} b_n \quad , \quad \lim_{n\to\infty}\frac{a_n}{b_n}=\frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}\ ,\ \lim_{n\to\infty} b_n\neq 0.$$

$$\lim_{x\to\infty}\left(\sqrt{x^2-5\cdot x+6}-x\right)=\lim_{x\to\infty}\frac{\sqrt{x^2-5\cdot x+6}-x}{1}=\begin{bmatrix}\text{racionalizacija}\\ \text{brojnika}\end{bmatrix}=$$

$$=\lim_{x\to\infty}\frac{\sqrt{x^2-5\cdot x+6}-x}{1}\cdot\frac{\sqrt{x^2-5\cdot x+6}+x}{\sqrt{x^2-5\cdot x+6}+x}=\lim_{x\to\infty}\frac{\left(\sqrt{x^2-5\cdot x+6}-x\right)\cdot\left(\sqrt{x^2-5\cdot x+6}+x\right)}{\sqrt{x^2-5\cdot x+6}+x}=$$

$$=\begin{bmatrix}\text{u brojniku je}\\ \text{razlika kvadrata}\end{bmatrix}=\lim_{x\to\infty}\frac{\left(\sqrt{x^2-5\cdot x+6}\right)^2-x^2}{\sqrt{x^2-5\cdot x+6}+x}=\lim_{x\to\infty}\frac{x^2-5\cdot x+6-x^2}{\sqrt{x^2-5\cdot x+6}+x}=$$

$$=\lim_{x\to\infty}\frac{x^2-5\cdot x+6-x^2}{\sqrt{x^2-5\cdot x+6}+x}=\lim_{x\to\infty}\frac{-5\cdot x+6}{\sqrt{x^2-5\cdot x+6}+x}=\begin{bmatrix}\text{brojnik i nazivnik}\\ \text{podijelimo sa } x\end{bmatrix}=$$

$$=\lim_{x\to\infty}\frac{\frac{-5\cdot x+6}{x}}{\frac{\sqrt{x^2-5\cdot x+6}+x}{x}}=\lim_{x\to\infty}\frac{\frac{-5\cdot x}{x}+\frac{6}{x}}{\frac{\sqrt{x^2-5\cdot x+6}}{x}+\frac{x}{x}}=\lim_{x\to\infty}\frac{\frac{-5\cdot x}{x}+\frac{6}{x}}{\frac{\sqrt{x^2-5\cdot x+6}}{x}+\frac{x}{x}}=$$

$$=\lim_{x\to\infty}\frac{-5+\frac{6}{x}}{\frac{\sqrt{x^2-5\cdot x+6}}{x}+1}=\lim_{x\to\infty}\frac{-5+\frac{6}{x}}{\sqrt{\frac{x^2-5\cdot x+6}{x^2}}+1}=\lim_{x\to\infty}\frac{-5+\frac{6}{x}}{\sqrt{\frac{x^2}{x^2}-\frac{5\cdot x}{x^2}+\frac{6}{x^2}}+1}=$$

$$=\lim_{x\to\infty}\frac{-5+\frac{6}{x}}{\sqrt{\frac{x^2}{x^2}-\frac{5\cdot x}{x^2}+\frac{6}{x^2}}+1}=\lim_{x\to\infty}\frac{-5+\frac{6}{x}}{\sqrt{1-\frac{5}{x}+\frac{6}{x^2}}+1}=\frac{-5+\lim_{x\to\infty}\frac{6}{x}}{\sqrt{1-\lim_{x\to\infty}\frac{5}{x}+\lim_{x\to\infty}\frac{6}{x^2}}+1}=$$

$$=\frac{-5+0}{\sqrt{1-0+0}+1}=\frac{-5}{\sqrt{1}+1}=\frac{-5}{1+1}=-\frac{5}{2}.$$

Vježba 110

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to \infty} \left(\sqrt{x^2 - 5 \cdot x + 2} - x \right)$.

Rezultat: $-\frac{5}{2}$.

Zadatak 111 (Lea, gimnazija)

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to \infty} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$.

Rješenje 111

Ponovimo!

$$(a-b) \cdot (a+b) = a^2 - b^2 \quad , \quad \left(\sqrt{a}\right)^2 = a.$$

Zakon distribucije množenja prema zbrajanju.

$$a \cdot (b+c) = a \cdot b + a \cdot c \quad , \quad a \cdot b + a \cdot c = a \cdot (b+c).$$

Važno je zapamtiti sljedeći limes

$$\lim_{n \to \infty} \frac{1}{n} = 0.$$

Limes iracionalnog izraza često se računa i tako da prebacimo iracionalnost iz brojnika u nazivnik (da se riješimo korijena u brojniku) ili obrnuto iz nazivnika u brojnik (da se riješimo korijena u nazivniku).

Ako su nizovi { a_n } i { b_n } konvergentni, tada vrijede sljedeća pravila o graničnoj vrijednosti (limesu):

$$\lim_{n \to \infty} c = c \;,\; c - \text{konstanta} \quad , \quad \lim_{n \to \infty} \frac{c}{n} = 0 \;,\; c - \text{konstanta} \quad , \quad \lim_{n \to \infty} \frac{1}{n} = 0$$

$$\lim_{n \to \infty} (c \cdot a_n) = c \cdot \lim_{n \to \infty} a_n \;,\; c - \text{konstanta} \quad , \quad \lim_{n \to \infty} (a_n \pm b_n) = \lim_{n \to \infty} a_n \pm \lim_{n \to \infty} b_n.$$

$$\lim_{n \to \infty} (a_n \cdot b_n) = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n \quad , \quad \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n \to \infty} a_n}{\lim\limits_{n \to \infty} b_n} \;,\; \lim_{n \to \infty} b_n \neq 0.$$

$$\lim_{x \to \infty} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \begin{bmatrix} \text{racionalizacija} \\ \text{brojnika} \end{bmatrix} = \lim_{x \to \infty} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} \cdot \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} =$$

$$= \lim_{x \to \infty} \frac{\left(\sqrt{1+x} - \sqrt{1-x}\right) \cdot \left(\sqrt{1+x} + \sqrt{1-x}\right)}{x \cdot \left(\sqrt{1+x} + \sqrt{1-x}\right)} = \begin{bmatrix} \text{u brojniku je} \\ \text{razlika kvadrata} \end{bmatrix} = \lim_{x \to \infty} \frac{\left(\sqrt{1+x}\right)^2 - \left(\sqrt{1-x}\right)^2}{x \cdot \left(\sqrt{1+x} + \sqrt{1-x}\right)} =$$

$$= \lim_{x \to \infty} \frac{1 + x - (1-x)}{x \cdot \left(\sqrt{1+x} + \sqrt{1-x}\right)} = \lim_{x \to \infty} \frac{1 + x - 1 + x}{x \cdot \left(\sqrt{1+x} + \sqrt{1-x}\right)} = \lim_{x \to \infty} \frac{\cancel{1} + x \cancel{-1} + x}{x \cdot \left(\sqrt{1+x} + \sqrt{1-x}\right)} =$$

$$= \lim_{x \to \infty} \frac{2 \cdot x}{x \cdot \left(\sqrt{1+x} + \sqrt{1-x}\right)} = \lim_{x \to \infty} \frac{2 \cdot \cancel{x}}{\cancel{x} \cdot \left(\sqrt{1+x} + \sqrt{1-x}\right)} = \lim_{x \to \infty} \frac{2}{\sqrt{1+x} + \sqrt{1-x}} = 0.$$

Vježba 111

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to \infty} \frac{\sqrt{2+x} - \sqrt{2-x}}{x}$.

Rezultat: 0.

Zadatak 112 (Lea, gimnazija)

Odredite graničnu vrijednost (limes): $\lim\limits_{n\to\infty}\frac{1-10^{n}}{1+10^{n+1}}$.

Rješenje 112

Ponovimo!

$$(a-b)\cdot(a+b)=a^{2}-b^{2}\quad,\quad\left(\sqrt{a}\right)^{2}=a\quad,\quad a^{n}\cdot a^{m}=a^{n+m}\quad,\quad a^{1}=a.$$

Zakon distribucije množenja prema zbrajanju.

$$a\cdot(b+c)=a\cdot b+a\cdot c\quad,\quad a\cdot b+a\cdot c=a\cdot(b+c).$$

Važno je zapamtiti sljedeći limes

$$\lim_{n\to\infty}\frac{1}{n}=0.$$

Limes iracionalnog izraza često se računa i tako da prebacimo iracionalnost iz brojnika u nazivnik (da se riješimo korijena u brojniku) ili obrnuto iz nazivnika u brojnik (da se riješimo korijena u nazivniku).

Ako je funkcija y = f(x) neprekidna u točki a, onda je limes funkcije u točki a jednak funkciji limesa nezavisne varijable x u točki a:

$$\lim_{x\to a}f(x)=f\left(\lim_{x\to a}x\right).$$

Ako postoje $\lim\limits_{x\to a}f(x)$ i $\lim\limits_{x\to a}g(x)$, tada vrijedi poučak:

$$\lim_{x\to a}\left[f(x)+g(x)\right]=\lim_{x\to a}f(x)+\lim_{x\to a}g(x).$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a}f(x)}{\lim\limits_{x\to a}g(x)}\quad,\quad\lim_{x\to a}g(x)\neq0\quad,\quad\lim_{n\to\infty}c=c\ ,\ c-\text{konstanta}.$$

Ako su nizovi { a_n } i { b_n } konvergentni, tada vrijede sljedeća pravila o graničnoj vrijednosti (limesu):

$$\lim_{n\to\infty}c=c\ ,\ c-\text{konstanta}\quad,\quad\lim_{n\to\infty}(a_n\pm b_n)=\lim_{n\to\infty}a_n\pm\lim_{n\to\infty}b_n.$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{\lim\limits_{n\to\infty}a_n}{\lim\limits_{n\to\infty}b_n}\quad,\quad\lim_{n\to\infty}b_n\neq0.$$

$$\lim_{n\to\infty}\frac{1-10^{n}}{1+10^{n+1}}=\lim_{n\to\infty}\frac{1-10^{n}}{1+10^{n}\cdot10^{1}}=\lim_{n\to\infty}\frac{1-10^{n}}{1+10^{n}\cdot10}=\lim_{n\to\infty}\frac{10^{n}\cdot\left(\frac{1}{10^{n}}-1\right)}{10^{n}\cdot\left(\frac{1}{10^{n}}+10\right)}=$$

$$=\lim_{n\to\infty}\frac{10^{n}\cdot\left(\frac{1}{10^{n}}-1\right)}{10^{n}\cdot\left(\frac{1}{10^{n}}+10\right)}=\lim_{n\to\infty}\frac{\frac{1}{10^{n}}-1}{\frac{1}{10^{n}}+10}=\frac{\lim\limits_{n\to\infty}\frac{1}{10^{n}}-1}{\lim\limits_{n\to\infty}\frac{1}{10^{n}}+10}=\frac{0-1}{0+10}=-\frac{1}{10}.$$

Vježba 112

Odredite graničnu vrijednost (limes): $\lim\limits_{n\to\infty}\frac{1+10^{n}}{1+10^{n+2}}$.

Rezultat: $-\frac{1}{100}$.

Zadatak 113 (Darko, gimnazija)

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to -1} \frac{x^3+1}{\sin(x+1)}$.

Rješenje 113

Ponovimo!

$$a^3+b^3=(a+b)\cdot\left(a^2-a\cdot b+b^2\right) \quad , \quad (a-b)^2=a^2-2\cdot a\cdot b+b^2 \quad , \quad n=\frac{n}{1}.$$

$$\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a\cdot d}{b\cdot c}.$$

Zakon distribucije množenja prema zbrajanju.

$$a\cdot(b+c)=a\cdot b+a\cdot c \quad , \quad a\cdot b+a\cdot c=a\cdot(b+c).$$

Važno je zapamtiti sljedeći limes

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x}=1.}.$$

Ako postoje $\lim\limits_{x \to a} f(x)$ i $\lim\limits_{x \to a} g(x)$, tada vrijedi poučak:

$$\lim_{x \to a}\left[f(x)\pm g(x)\right]=\lim_{x \to a} f(x)\pm\lim_{x \to a} g(x).$$

$$\lim_{x \to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} \ , \ \lim_{x \to a} g(x)\neq 0 \quad , \quad \lim_{n \to \infty} c=c \ , \ c-\text{konstanta}.$$

$$\lim_{x \to -1}\frac{x^3+1}{\sin(x+1)}=\lim_{x \to -1}\frac{(x+1)\cdot\left(x^2-x+1\right)}{\sin(x+1)}=\lim_{x \to -1}\frac{x^2-x+1}{\frac{\sin(x+1)}{x+1}}=\begin{bmatrix}\text{supstitucija}\\ t=x+1\\ x=t-1\\ x\to -1 \ \Rightarrow \ t\to 0\end{bmatrix}=$$

$$=\lim_{t \to 0}\frac{(t-1)^2-(t-1)+1}{\frac{\sin t}{t}}=\lim_{t \to 0}\frac{t^2-2\cdot t+1-t+1+1}{\frac{\sin t}{t}}=\lim_{t \to 0}\frac{t^2-3\cdot t+3}{\frac{\sin t}{t}}=$$

$$=\frac{\lim\limits_{t \to 0}\left(t^2-3\cdot t+3\right)}{\lim\limits_{t \to 0}\frac{\sin t}{t}}=\frac{0^2-3\cdot 0+3}{1}=\frac{3}{1}=3.$$

Vježba 113

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to 1} \frac{x^3-1}{\sin(x-1)}$.

Rezultat: 3.

Zadatak 114 (Nebojša, gimnazija)

Odredite graničnu vrijednost (limes): $\lim\limits_{x \to 8} \frac{\sqrt{x+8}-4}{2-\sqrt[3]{x}}$.

Rješenje 114

Ponovimo!

$$a^2-b^2=(a-b)\cdot(a+b) \quad , \quad \left(\sqrt[n]{a}\right)^n=a \quad , \quad \sqrt[n]{a^n}=a \ , \ a\geq 0.$$

$$a^3-b^3=(a-b)\cdot\left(a^2+a\cdot b+b^2\right).$$

Zakon distribucije množenja prema zbrajanju.

$$a\cdot(b+c)=a\cdot b+a\cdot c \quad , \quad a\cdot b+a\cdot c=a\cdot(b+c).$$

Limes iracionalnog izraza često se računa i tako da prebacimo iracionalnost iz brojnika u nazivnik (da se riješimo korijena u brojniku) ili obrnuto iz nazivnika u brojnik (da se riješimo korijena u nazivniku).

Ako postoje $\lim\limits_{x\to a} f(x)$ i $\lim\limits_{x\to a} g(x)$, tada vrijedi poučak:

$$\lim_{x\to a} c\cdot f(x)=c\cdot\lim_{x\to a} f(x) \ , \ c \text{ konstanta} \quad , \quad \lim_{x\to a}\left[f(x)+g(x)\right]=\lim_{x\to a} f(x)+\lim_{x\to a} g(x).$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} \ , \ \lim_{x\to a} g(x)\neq 0 \quad , \quad \lim_{n\to\infty} c=c \ , \ c-\text{konstanta}.$$

Važno je zapamtiti sljedeći limes

$$\lim_{x\to\infty}\frac{1}{x}=0.$$

$$\lim_{x\to 8}\frac{\sqrt{x+8}-4}{2-\sqrt[3]{x}}=\left[\begin{array}{l}\text{racionalizacija brojnika}\\ \text{pomoću razlike kvadrata}\end{array}\right]=\lim_{x\to 8}\frac{\sqrt{x+8}-4}{2-\sqrt[3]{x}}\cdot\frac{\sqrt{x+8}+4}{\sqrt{x+8}+4}=$$

$$=\lim_{x\to 8}\frac{\left(\sqrt{x+8}-4\right)\cdot\left(\sqrt{x+8}+4\right)}{\left(2-\sqrt[3]{x}\right)\cdot\left(\sqrt{x+8}+4\right)}=\lim_{x\to 8}\frac{\left(\sqrt{x+8}\right)^2-4^2}{\left(2-\sqrt[3]{x}\right)\cdot\left(\sqrt{x+8}+4\right)}=\lim_{x\to 8}\frac{x+8-16}{\left(2-\sqrt[3]{x}\right)\cdot\left(\sqrt{x+8}+4\right)}=$$

$$=\lim_{x\to 8}\frac{x-8}{\left(2-\sqrt[3]{x}\right)\cdot\left(\sqrt{x+8}+4\right)}=\lim_{x\to 8}\frac{x-8}{\left(\sqrt{x+8}+4\right)\cdot\left(2-\sqrt[3]{x}\right)}=$$

$$=\left[\begin{array}{l}\text{racionalizacija nazivnika}\\ \text{pomoću razlike kubova}\end{array}\right]=\lim_{x\to 8}\frac{x-8}{\left(\sqrt{x+8}+4\right)\cdot\left(2-\sqrt[3]{x}\right)}\cdot\frac{2^2+2\cdot\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2}{2^2+2\cdot\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2}=$$

$$=\lim_{x\to 8}\frac{(x-8)\cdot\left(2^2+2\cdot\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2\right)}{\left(\sqrt{x+8}+4\right)\cdot\left(2-\sqrt[3]{x}\right)\cdot\left(2^2+2\cdot\sqrt[3]{x}+\left(\sqrt[3]{x}\right)^2\right)}=\lim_{x\to 8}\frac{(x-8)\cdot\left(4+2\cdot\sqrt[3]{x}+\sqrt[3]{x^2}\right)}{\left(\sqrt{x+8}+4\right)\cdot\left(2^3-\left(\sqrt[3]{x}\right)^3\right)}=$$

$$=\lim_{x\to 8}\frac{(x-8)\cdot\left(4+2\cdot\sqrt[3]{x}+\sqrt[3]{x^2}\right)}{\left(\sqrt{x+8}+4\right)\cdot(8-x)}=\lim_{x\to 8}\frac{-(8-x)\cdot\left(4+2\cdot\sqrt[3]{x}+\sqrt[3]{x^2}\right)}{\left(\sqrt{x+8}+4\right)\cdot(8-x)}=$$

$$=\lim_{x\to 8}\frac{-(8-x)\cdot\left(4+2\cdot\sqrt[3]{x}+\sqrt[3]{x^2}\right)}{\left(\sqrt{x+8}+4\right)\cdot(8-x)}=-\lim_{x\to 8}\frac{4+2\cdot\sqrt[3]{x}+\sqrt[3]{x^2}}{\sqrt{x+8}+4}=$$

$$=-\frac{\lim\limits_{x\to 8}\left(4+2\cdot\sqrt[3]{x}+\sqrt[3]{x^2}\right)}{\lim\limits_{x\to 8}\left(\sqrt{x+8}+4\right)}=-\frac{4+2\cdot\sqrt[3]{8}+\sqrt[3]{8^2}}{\sqrt{8+8}+4}=-\frac{4+2\cdot\sqrt[3]{8}+\sqrt[3]{64}}{\sqrt{16}+4}=$$

$$=-\frac{4+2\cdot\sqrt[3]{2^3}+\sqrt[3]{4^3}}{\sqrt{4^2}+4}=-\frac{4+2\cdot 2+4}{4+4}=-\frac{12}{8}=-\frac{12}{8}=-\frac{3}{2}.$$

Vježba 114

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 8}\frac{2-\sqrt[3]{x}}{\sqrt{x+8}-4}.$

Rezultat: $-\frac{2}{3}.$

Zadatak 115 (Nebojša, gimnazija)

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to\infty}\frac{\sqrt[4]{x^2-2\cdot x+3}}{x^2+3\cdot x}.$

Rješenje 115

Ponovimo!

Proširiti razlomak znači brojnik i nazivnik tog razlomka pomnožiti istim brojem različitim od nule i jedinice

$$\frac{a}{b}=\frac{a\cdot n}{b\cdot n}\ ,\ n\neq 0\ ,\ n\neq 1.$$

$$\sqrt[n]{a^m}=\sqrt[n\cdot p]{a^{m\cdot p}}\quad ,\quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}}\quad ,\quad \frac{a+b}{n}=\frac{a}{n}+\frac{b}{n}\quad ,\quad \frac{a-b}{n}=\frac{a}{n}-\frac{b}{n}.$$

$$\frac{a^n}{a^m}=a^{n-m}.$$

Ako postoje $\lim\limits_{x\to a} f(x)$ i $\lim\limits_{x\to a} g(x)$, tada vrijedi poučak:

$$\lim_{x\to a} c\cdot f(x)=c\cdot\lim_{x\to a} f(x)\ ,\ c\ \text{konstanta}\quad ,\quad \lim_{x\to a}\left[f(x)+g(x)\right]=\lim_{x\to a} f(x)+\lim_{x\to a} g(x).$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}\ ,\ \lim_{x\to a} g(x)\neq 0\quad ,\quad \lim_{n\to\infty} c=c\ ,\ c-\text{konstanta}.$$

Važno je zapamtiti sljedeći limes

$$\lim_{x\to\infty}\frac{1}{x}=0.$$

Ako je funkcija y = f(x) neprekidna u točki a, onda je limes funkcije u točki a jednak funkciji limesa nezavisne varijable x u točki a:

$$\lim_{x\to a} f(x)=f\left(\lim_{x\to a} x\right).$$

$$\lim_{x\to\infty}\frac{\sqrt[4]{x^2-2\cdot x+3}}{x^2+3\cdot x}=\left[\begin{array}{l}\text{brojnik i nazivnik podijelimo sa } x^2\text{, tj.}\\ \text{razlomak proširimo sa } \frac{1}{x^2}\end{array}\right]=\lim_{x\to\infty}\frac{\frac{\sqrt[4]{x^2-2\cdot x+3}}{x^2}}{\frac{x^2+3\cdot x}{x^2}}=$$

$$=\lim_{x\to\infty}\frac{\frac{\sqrt[4]{x^2-2\cdot x+3}}{\sqrt[4]{x^8}}}{\frac{x^2+3\cdot x}{x^2}}=\lim_{x\to\infty}\frac{\sqrt[4]{\frac{x^2-2\cdot x+3}{x^8}}}{\frac{x^2+3\cdot x}{x^2}}=\lim_{x\to\infty}\frac{\sqrt[4]{\frac{x^2}{x^8}-2\cdot\frac{x}{x^8}+\frac{3}{x^8}}}{\frac{x^2}{x^2}+3\cdot\frac{x}{x^2}}=$$

$$=\lim_{x\to\infty}\frac{\sqrt[4]{\frac{x^2}{x^8}-2\cdot\frac{x}{x^8}+\frac{3}{x^8}}}{\frac{x^2}{x^2}+3\cdot\frac{x}{x^2}}=\lim_{x\to\infty}\frac{\sqrt[4]{\frac{1}{x^6}-2\cdot\frac{1}{x^7}+\frac{3}{x^8}}}{1+3\cdot\frac{1}{x}}=\frac{\lim\limits_{x\to\infty}\sqrt[4]{\frac{1}{x^6}-2\cdot\frac{1}{x^7}+\frac{3}{x^8}}}{\lim\limits_{x\to\infty}\left(1+3\cdot\frac{1}{x}\right)}=$$

$$=\frac{\sqrt[4]{\lim\limits_{x\to\infty}\frac{1}{x^6}-2\cdot\lim\limits_{x\to\infty}\frac{1}{x^7}+3\cdot\lim\limits_{x\to\infty}\frac{1}{x^8}}}{1+3\cdot\lim\limits_{x\to\infty}\frac{1}{x}}=\frac{\sqrt[4]{0-2\cdot 0+3\cdot 0}}{1+3\cdot 0}=\frac{\sqrt[4]{0}}{1}=\frac{0}{1}=0.$$

Vježba 115

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to\infty}\frac{\sqrt[4]{x^2-2\cdot x+7}}{x^2+5\cdot x}$.

Rezultat: 0.

Zadatak 116 (Ana, gimnazija)

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 0}\frac{x-\sqrt{x}}{\sqrt{x}}$.

Rješenje 116

Ponovimo!

$$\left(\sqrt[n]{a}\right)^n=a\quad,\quad\frac{a-b}{n}=\frac{a}{n}-\frac{b}{n}.$$

Zakon distribucije množenja prema zbrajanju.

$$a\cdot(b+c)=a\cdot b+a\cdot c\quad,\quad a\cdot b+a\cdot c=a\cdot(b+c).$$

Skratiti razlomak znači brojnik i nazivnik tog razlomka podijeliti istim brojem različitim od nule i jedinice

$$\frac{a\cdot n}{b\cdot n}=\frac{a}{b}\ ,\ n\neq 0\ ,\ n\neq 1.$$

Ako postoje $\lim\limits_{x\to a} f(x)$ i $\lim\limits_{x\to a} g(x)$, tada vrijedi poučak:

$$\lim_{x\to a}\left[f(x)-g(x)\right]=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)\quad,\quad\lim_{n\to\infty} c=c\ ,\ c-\text{konstanta}.$$

Ako je funkcija y = f(x) neprekidna u točki a, onda je limes funkcije u točki a jednak funkciji limesa nezavisne varijable x u točki a:

$$\lim_{x\to a} f(x) = f\left(\lim_{x\to a} x\right).$$

Limes iracionalnog izraza često se računa i tako da prebacimo iracionalnost iz brojnika u nazivnik (da se riješimo korijena u brojniku) ili obrnuto iz nazivnika u brojnik (da se riješimo korijena u nazivniku).

1.inačica

$$\lim_{x\to 0} \frac{x-\sqrt{x}}{\sqrt{x}} = \lim_{x\to 0} \frac{\left(\sqrt{x}\right)^2-\sqrt{x}}{\sqrt{x}} = \lim_{x\to 0} \frac{\sqrt{x}\cdot\left(\sqrt{x}-1\right)}{\sqrt{x}} = \lim_{x\to 0} \frac{\cancel{\sqrt{x}}\cdot\left(\sqrt{x}-1\right)}{\cancel{\sqrt{x}}} =$$

$$= \lim_{x\to 0} \left(\sqrt{x}-1\right) = \sqrt{0}-1 = 0-1 = -1.$$

2.inačica

$$\lim_{x\to 0} \frac{x-\sqrt{x}}{\sqrt{x}} = \lim_{x\to 0} \left(\frac{x}{\sqrt{x}}-\frac{\sqrt{x}}{\sqrt{x}}\right) = \lim_{x\to 0} \left(\frac{\left(\sqrt{x}\right)^2}{\sqrt{x}}-\frac{\sqrt{x}}{\sqrt{x}}\right) = \lim_{x\to 0} \left(\frac{\left(\sqrt{x}\right)^{2}}{\sqrt{x}}-\frac{\sqrt{x}}{\sqrt{x}}\right) =$$

$$= \lim_{x\to 0} \left(\sqrt{x}-1\right) = \sqrt{0}-1 = 0-1 = -1.$$

3.inačica

$$\lim_{x\to 0} \frac{x-\sqrt{x}}{\sqrt{x}} = \left[\begin{array}{l}\text{racionalizacija}\\ \text{nazivnika}\end{array}\right] = \lim_{x\to 0} \frac{x-\sqrt{x}}{\sqrt{x}}\cdot\frac{\sqrt{x}}{\sqrt{x}} = \lim_{x\to 0} \frac{\left(x-\sqrt{x}\right)\cdot\sqrt{x}}{\left(\sqrt{x}\right)^2} = \lim_{x\to 0} \frac{x\cdot\sqrt{x}-\left(\sqrt{x}\right)^2}{\left(\sqrt{x}\right)^2} =$$

$$= \lim_{x\to 0} \frac{x\cdot\sqrt{x}-x}{x} = \lim_{x\to 0} \frac{x\cdot\left(\sqrt{x}-1\right)}{x} = \lim_{x\to 0} \frac{\cancel{x}\cdot\left(\sqrt{x}-1\right)}{\cancel{x}} = \lim_{x\to 0} \left(\sqrt{x}-1\right) = \sqrt{0}-1 = 0-1 = -1.$$

Vježba 116

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 0} \dfrac{\sqrt{x}}{x-\sqrt{x}}.$

Rezultat: $-1.$

Zadatak 117 (Ana, gimnazija)

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+1}}{x+1}.$

Rješenje 117

Ponovimo!

$$\left(\sqrt[n]{a}\right)^n = a \quad , \quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \quad , \quad \frac{a+b}{n} = \frac{a}{n}+\frac{b}{n} \quad , \quad (a+b)^2 = a^2+2\cdot a\cdot b+b^2.$$

$$\sqrt{a^2} = a \ , \ a \ge 0.$$

Proširiti razlomak znači brojnik i nazivnik tog razlomka pomnožiti istim brojem različitim od nule i jedinice

$$\frac{a}{b} = \frac{a\cdot n}{b\cdot n} \ , \ n \ne 0 \ , \ n \ne 1.$$

Skratiti razlomak znači brojnik i nazivnik tog razlomka podijeliti istim brojem različitim od nule i jedinice

$$\frac{a\cdot n}{b\cdot n}=\frac{a}{b}\ ,\ n\neq 0\ ,\ n\neq 1.$$

Važno je zapamtiti sljedeći limes

$$\lim_{x\to\infty}\frac{1}{x}=0.$$

Ako je funkcija y = f(x) neprekidna u točki a, onda je limes funkcije u točki a jednak funkciji limesa nezavisne varijable x u točki a:

$$\lim_{x\to a} f(x)=f\left(\lim_{x\to a} x\right).$$

Ako postoje $\lim_{x\to a} f(x)$ i $\lim_{x\to a} g(x)$, tada vrijedi poučak:

$$\lim_{x\to a}\left[f(x)+g(x)\right]=\lim_{x\to a} f(x)+\lim_{x\to a} g(x).$$

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}\ ,\ \lim_{x\to a} g(x)\neq 0\quad ,\quad \lim_{n\to\infty} c=c\ ,\ c-\text{konstanta}.$$

1.inačica

$$\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{x+1}=\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{\sqrt{(x+1)^2}}=\lim_{x\to\infty}\sqrt{\frac{x^2+1}{(x+1)^2}}=\lim_{x\to\infty}\sqrt{\frac{x^2+1}{x^2+2\cdot x+1}}=$$

$$=\lim_{x\to\infty}\sqrt{\frac{x^2+1}{x^2+2\cdot x+1}}=\left[\begin{array}{l}\text{brojnik i nazivnik podijelimo sa } x^2\text{, tj.}\\ \text{razlomak proširimo sa } \frac{1}{x^2}\end{array}\right]=\lim_{x\to\infty}\sqrt{\frac{\frac{x^2+1}{x^2}}{\frac{x^2+2\cdot x+1}{x^2}}}=$$

$$=\lim_{x\to\infty}\sqrt{\frac{\frac{x^2}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{2\cdot x}{x^2}+\frac{1}{x^2}}}=\lim_{x\to\infty}\sqrt{\frac{\frac{x^2}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{2\cdot x}{x^2}+\frac{1}{x^2}}}=\lim_{x\to\infty}\sqrt{\frac{1+\frac{1}{x^2}}{1+\frac{2}{x}+\frac{1}{x^2}}}=$$

$$=\sqrt{\lim_{x\to\infty}\frac{1+\frac{1}{x^2}}{1+\frac{2}{x}+\frac{1}{x^2}}}=\sqrt{\frac{\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)}{\lim_{x\to\infty}\left(1+\frac{2}{x}+\frac{1}{x^2}\right)}}=\sqrt{\frac{1+0}{1+0+0}}=\sqrt{\frac{1}{1}}=\sqrt{1}=1.$$

2.inačica

$$\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{x+1}=\left[\begin{array}{l}\text{brojnik i nazivnik podijelimo sa } x\text{, tj.}\\ \text{razlomak proširimo sa } \frac{1}{x}\end{array}\right]=\lim_{x\to\infty}\frac{\frac{\sqrt{x^2+1}}{x}}{\frac{x+1}{x}}=\lim_{x\to\infty}\frac{\frac{\sqrt{x^2+1}}{\sqrt{x^2}}}{\frac{x+1}{x}}=$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{x^2+1}{x^2}}}{\frac{x+1}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}}{\frac{x}{x}+\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}}{\frac{x}{x}+\frac{1}{x}} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x^2}}}{1+\frac{1}{x}} =$$

$$= \frac{\lim_{x\to\infty} \sqrt{1+\frac{1}{x^2}}}{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)} = \frac{\sqrt{\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)}}{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)} = \frac{\sqrt{1+0}}{1+0} = \frac{\sqrt{1}}{1} = 1.$$

Vježba 117

Odredite graničnu vrijednost (limes): $\lim_{x\to\infty} \frac{x+1}{\sqrt{x^2+1}}$.

Rezultat: 1.

Zadatak 118 (Mira, gimnazija)

Odredite graničnu vrijednost (limes): $\lim_{a\to 0} \frac{\sqrt{b^2-4\cdot a\cdot c}-b}{2\cdot a}$.

Rješenje 118

Ponovimo!

$$\left(\sqrt[n]{a}\right)^n = a \quad , \quad \sqrt{a^2} = a \ , \ a \ge 0 \quad , \quad a^2-b^2 = (a-b)\cdot(a+b).$$

Proširiti razlomak znači brojnik i nazivnik tog razlomka pomnožiti istim brojem različitim od nule i jedinice

$$\frac{a}{b} = \frac{a\cdot n}{b\cdot n} \ , \ n \ne 0 \ , \ n \ne 1.$$

Skratiti razlomak znači brojnik i nazivnik tog razlomka podijeliti istim brojem različitim od nule i jedinice

$$\frac{a\cdot n}{b\cdot n} = \frac{a}{b} \ , \ n \ne 0 \ , \ n \ne 1.$$

Ako je funkcija y = f(x) neprekidna u točki a, onda je limes funkcije u točki a jednak funkciji limesa nezavisne varijable x u točki a:

$$\lim_{x\to a} f(x) = f\left(\lim_{x\to a} x\right).$$

Ako postoje $\lim_{x\to a} f(x)$ i $\lim_{x\to a} g(x)$, tada vrijedi poučak:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \ , \ \lim_{x\to a} g(x) \ne 0 \quad , \quad \lim_{n\to\infty} c = c \ , \ c - \text{konstanta}.$$

Limes iracionalnog izraza često se računa i tako da prebacimo iracionalnost iz brojnika u nazivnik (da se riješimo korijena u brojniku) ili obrnuto iz nazivnika u brojnik (da se riješimo korijena u nazivniku).

$$\lim_{a\to 0} \frac{\sqrt{b^2-4\cdot a\cdot c}-b}{2\cdot a} = \left[\begin{array}{l}\text{racionalizacija}\\ \text{brojnika}\end{array}\right] = \lim_{a\to 0} \frac{\sqrt{b^2-4\cdot a\cdot c}-b}{2\cdot a} \cdot \frac{\sqrt{b^2-4\cdot a\cdot c}+b}{\sqrt{b^2-4\cdot a\cdot c}+b} =$$

$$= \lim_{a \to 0} \frac{\left(\sqrt{b^2 - 4 \cdot a \cdot c} - b\right) \cdot \left(\sqrt{b^2 - 4 \cdot a \cdot c} + b\right)}{2 \cdot a \cdot \left(\sqrt{b^2 - 4 \cdot a \cdot c} + b\right)} = \lim_{a \to 0} \frac{\left(\sqrt{b^2 - 4 \cdot a \cdot c}\right)^2 - b^2}{2 \cdot a \cdot \left(\sqrt{b^2 - 4 \cdot a \cdot c} + b\right)} =$$

$$= \lim_{a \to 0} \frac{b^2 - 4 \cdot a \cdot c - b^2}{2 \cdot a \cdot \left(\sqrt{b^2 - 4 \cdot a \cdot c} + b\right)} = \lim_{a \to 0} \frac{b^2 - 4 \cdot a \cdot c - b^2}{2 \cdot a \cdot \left(\sqrt{b^2 - 4 \cdot a \cdot c} + b\right)} = \lim_{a \to 0} \frac{-4 \cdot a \cdot c}{2 \cdot a \cdot \left(\sqrt{b^2 - 4 \cdot a \cdot c} + b\right)} =$$

$$= \lim_{a \to 0} \frac{-4 \cdot a \cdot c}{2 \cdot a \cdot \left(\sqrt{b^2 - 4 \cdot a \cdot c} + b\right)} = \lim_{a \to 0} \frac{-2 \cdot c}{\sqrt{b^2 - 4 \cdot a \cdot c} + b} = \frac{\lim_{a \to 0} (-2 \cdot c)}{\lim_{a \to 0} \left(\sqrt{b^2 - 4 \cdot a \cdot c} + b\right)} =$$

$$= \frac{-2 \cdot c}{\sqrt{b^2 - 4 \cdot 0 \cdot c} + b} = \frac{-2 \cdot c}{\sqrt{b^2} + b} = \frac{-2 \cdot c}{b + b} = \frac{-2 \cdot c}{2 \cdot b} = \frac{-2 \cdot c}{2 \cdot b} = -\frac{c}{b}.$$

Vježba 118

Odredite graničnu vrijednost (limes): $\lim_{a \to 0} \frac{2 \cdot a}{\sqrt{b^2 - 4 \cdot a \cdot c} - b}$.

Rezultat: $-\frac{b}{c}.$

Zadatak 119 (Branko, srednja škola)

Odredite graničnu vrijednost (limes):

$$\lim_{n \to \infty} \frac{1 + a + a^2 + a^3 + ... + a^n}{1 + b + b^2 + b^3 + ... + b^n}, \text{ gdje je } |a| < 1, |b| < 1.$$

Rješenje 119

Ponovimo!

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a \cdot d}{b \cdot c} \quad , \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.$$

Niz (a_n) je geometrijski niz ako je svaki član niza, počevši od drugog, jednak prethodnom članu pomnoženom s konstantom $q \neq 0$, tj.

$$a_{n+1} = a_n \cdot q.$$

Broj q naziva se količnik (kvocijent) geometrijskog niza.
Niz je geometrijski ako je omjer svakog člana i člana ispred njega stalan:

$$\frac{a_n}{a_{n-1}} = q.$$

Zbroj prvih n članova geometrijskog niza je

$$s_n = a_1 \cdot \frac{q^n - 1}{q - 1} \ , \ q \neq 1.$$

Ako su nizovi $\{a_n\}$ i $\{b_n\}$ konvergentni, tada vrijede sljedeća pravila o graničnoj vrijednosti (limesu):

$$\lim_{n \to \infty} c = c, \ c \text{ je konstanta} \quad , \quad \lim_{n \to \infty} (c \cdot a_n) = c \cdot \lim_{n \to \infty} a_n \ , \ c \text{ je konstanta}.$$

$$\lim_{n \to \infty} (a_n \pm b_n) = \lim_{n \to \infty} a_n \pm \lim_{n \to \infty} b_n.$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{\lim\limits_{n\to\infty}a_n}{\lim\limits_{n\to\infty}b_n}\ ,\ \lim_{n\to\infty}b_n\neq 0..$$

$$\lim_{n\to\infty}q^n=0\ ,\ |q|<1.$$

$$\lim_{n\to\infty}\frac{1+a+a^2+a^3+...+a^n}{1+b+b^2+b^3+...+b^n}=\left[\begin{array}{l}\text{u brojniku je geometrijski red sa količnikom } a\\ \text{u nazivniku je geometrijski red sa količnikom } b\end{array}\right]=$$

$$=\lim_{n\to\infty}\frac{1\cdot\frac{a^{n+1}-1}{a-1}}{1\cdot\frac{b^{n+1}-1}{b-1}}=\lim_{n\to\infty}\frac{\frac{a^{n+1}-1}{a-1}}{\frac{b^{n+1}-1}{b-1}}=\lim_{n\to\infty}\frac{(b-1)\cdot\left(a^{n+1}-1\right)}{(a-1)\cdot\left(b^{n+1}-1\right)}=$$

$$=\frac{b-1}{a-1}\cdot\lim_{n\to\infty}\frac{a^{n+1}-1}{b^{n+1}-1}=\frac{b-1}{a-1}\cdot\frac{\lim\limits_{n\to\infty}\left(a^{n+1}-1\right)}{\lim\limits_{n\to\infty}\left(b^{n+1}-1\right)}=\frac{b-1}{a-1}\cdot\frac{\lim\limits_{n\to\infty}a^{n+1}-1}{\lim\limits_{n\to\infty}b^{n+1}-1}=$$

$$=\frac{b-1}{a-1}\cdot\frac{0-1}{0-1}=\frac{b-1}{a-1}\cdot\frac{-1}{-1}=\frac{b-1}{a-1}\cdot\frac{1}{1}=\frac{b-1}{a-1}.$$

Vježba 119

Odredite graničnu vrijednost (limes):

$$\lim_{n\to\infty}\frac{1+b+b^2+b^3+...+b^n}{1+a+a^2+a^3+...+a^n},\ \text{gdje je } |a|<1,\ |b|<1.$$

Rezultat: $\frac{a-1}{b-1}.$

Zadatak 120 (Nina, elektrotehnička škola)

Odredite graničnu vrijednost (limes): $\lim\limits_{x\to 4}\frac{x^2-4\cdot x}{x^2-5\cdot x+4}.$

Rješenje 120

Ponovimo!

$$a^2-b^2=(a-b)\cdot(a+b).$$

Zakon distribucije množenja prema zbrajanju.

$$a\cdot(b+c)=a\cdot b+a\cdot c\quad ,\quad a\cdot b+a\cdot c=a\cdot(b+c).$$

Skratiti razlomak znači brojnik i nazivnik tog razlomka podijeliti istim brojem različitim od nule i jedinice

$$\frac{a\cdot n}{b\cdot n}=\frac{a}{b}\ ,\ n\neq 0\ ,\ n\neq 1.$$

Ako su P(x) i Q(x) cijeli polinomi, a P(a) = Q(a) = 0, tada se limes racionalnog razlomka

$$\lim_{x\to a}\frac{P(x)}{Q(x)}$$

dobije tako da razlomak $\frac{P(x)}{Q(x)}$ skratimo jedanput ili nekoliko puta s binomom x – a.

Ponovimo L' Hospitalovo pravilo [čitaj: Lopitál].

Neka zadana funkcija ima oblik količnika $f(x)=\frac{u(x)}{v(x)}$ i neka je za x = a ona jednaka

$f(a)=\frac{u(a)}{v(a)}=\frac{0}{0}$ ili $f(a)=\frac{u(a)}{v(a)}=\frac{\infty}{\infty}$, tada je prava vrijednost funkcije f(x) za x = a, tj.

$$f(a)=\lim_{x\to a}\frac{u(x)}{v(x)}=\lim_{x\to a}\frac{u'(x)}{v'(x)}=\lim_{x\to a}\frac{u''(x)}{v''(x)}=\lim_{x\to a}\frac{u'''(x)}{v'''(x)}=\ldots \text{ itd,}$$

dok ne dođemo do određene vrijednosti konačne ili beskonačne. Derivira se posebno brojnik i posebno nazivnik, pri čemu se iza svakog deriviranja dobiveni izraz uredi i uvrsti zadani x = a. Zadatak je riješen, čim dobijemo u brojniku ili u nazivniku konačnu ili beskonačnu vrijednost.

1.inačica

$$\lim_{x\to 4}\frac{x^2-4\cdot x}{x^2-5\cdot x+4}=\begin{bmatrix}\text{u brojniku izlučimo } x,\\ \text{nazivnik rastavimo na faktore metodom grupiranja}\end{bmatrix}=\lim_{x\to 4}\frac{x\cdot(x-4)}{x^2-4\cdot x-x+4}=$$

$$=\lim_{x\to 4}\frac{x\cdot(x-4)}{\left(x^2-4\cdot x\right)+(-x+4)}=\begin{bmatrix}\text{u nazivniku iz prve zagrade izlučimo } x,\\ \text{iz druge zagrade izlučimo } -1\end{bmatrix}=$$

$$=\lim_{x\to 4}\frac{x\cdot(x-4)}{x\cdot(x-4)-1\cdot(x-4)}=\begin{bmatrix}\text{u nazivniku}\\ \text{izlučimo } x-4\end{bmatrix}=\lim_{x\to 4}\frac{x\cdot(x-4)}{x\cdot(x-4)-1\cdot(x-4)}=$$

$$=\lim_{x\to 4}\frac{x\cdot(x-4)}{(x-4)\cdot(x-1)}=\lim_{x\to 4}\frac{x\cdot\cancel{(x-4)}}{\cancel{(x-4)}\cdot(x-1)}=\lim_{x\to 4}\frac{x}{x-1}=\frac{4}{4-1}=\frac{4}{3}.$$

2.inačica

$$\lim_{x\to 4}\frac{x^2-4\cdot x}{x^2-5\cdot x+4}=\begin{bmatrix}\text{u brojniku izlučimo } x,\\ \text{nazivnik rastavimo na faktore metodom grupiranja}\end{bmatrix}=\lim_{x\to 4}\frac{x\cdot(x-4)}{x^2-5\cdot x+5-1}=$$

$$=\lim_{x\to 4}\frac{x\cdot(x-4)}{x^2-1-5\cdot x+5}=\lim_{x\to 4}\frac{x\cdot(x-4)}{\left(x^2-1\right)+(-5\cdot x+5)}=$$

$$=\begin{bmatrix}\text{u nazivniku u prvoj zagradi prepoznajemo razliku kvadrata,}\\ \text{iz druge zagrade izlučimo } -5\end{bmatrix}=$$

$$=\lim_{x\to 4}\frac{x\cdot(x-4)}{(x-1)\cdot(x+1)-5\cdot(x-1)}=\begin{bmatrix}\text{u nazivniku}\\ \text{izlučimo } x-1\end{bmatrix}=\lim_{x\to 4}\frac{x\cdot(x-4)}{(x-1)\cdot(x+1)-5\cdot(x-1)}=$$

$$=\lim_{x\to 4}\frac{x\cdot(x-4)}{(x-1)\cdot(x+1-5)}=\lim_{x\to 4}\frac{x\cdot(x-4)}{(x-1)\cdot(x-4)}=\lim_{x\to 4}\frac{x\cdot\cancel{(x-4)}}{(x-1)\cdot\cancel{(x-4)}}=\lim_{x\to 4}\frac{x}{x-1}=\frac{4}{4-1}=\frac{4}{3}.$$

3.inačica

Računamo pomoću L' Hospitalovog pravila.

$$\lim_{x\to 4}\frac{x^2-4\cdot x}{x^2-5\cdot x+4}=\begin{bmatrix}x=4\ \Rightarrow\ \frac{4^2-4\cdot 4}{4^2-5\cdot 4+4}=\frac{16-16}{16-20+4}=\frac{0}{0}\\ \text{deriviramo posebno brojnik, posebno nazivnik}\end{bmatrix}=$$

$$= \lim_{x\to 4}\frac{\left(x^2-4\cdot x\right)'}{\left(x^2-5\cdot x+4\right)'} = \lim_{x\to 4}\frac{\left(x^2\right)'-4\cdot x'}{\left(x^2\right)'-5\cdot x'+4'} = \lim_{x\to 4}\frac{2\cdot x-4\cdot 1}{2\cdot x-5\cdot 1+0} = \lim_{x\to 4}\frac{2\cdot x-4}{2\cdot x-5} =$$

$$= \left[\begin{array}{c} x=4 \;\Rightarrow\; \dfrac{2\cdot 4-4}{2\cdot 4-5}=\dfrac{8-4}{8-5}=\dfrac{4}{3} \\ \text{zadatak je riješen} \end{array}\right] = \frac{2\cdot 4-4}{2\cdot 4-5} = \frac{8-4}{8-5} = \frac{4}{3}.$$

Vježba 120

Odredite graničnu vrijednost (limes): $\lim\limits_{n\to 4}\dfrac{x^2-4\cdot x}{x^2-16}$.

Rezultat: $\dfrac{1}{2}$.