Zadatak 021 (Ivana, gimnazija)

Dokažite poučak o derivaciji zbroja: (f + g) ' = f ' + g '.

Rješenje 021

Ponovimo!

Oznake za derivaciju su:

$$y'=\frac{dy}{dx}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=f'(x).$$

Stavimo da je

$$y(x)=f(x)+g(x).$$

Tada je

$$y(x+\Delta x)=f(x+\Delta x)+g(x+\Delta x).$$

Prirast funkcije iznosi:

$$\Delta y=y(x+\Delta x)-y(x)\Rightarrow\Delta y=\left[f(x+\Delta x)+g(x+\Delta x)\right]-\left[f(x)+g(x)\right]\Rightarrow$$

$$\Rightarrow\Delta y=f(x+\Delta x)+g(x+\Delta x)-f(x)-g(x)\Rightarrow\Delta y=\underbrace{f(x+\Delta x)-f(x)}_{\Delta f(x)}+\underbrace{g(x+\Delta x)-g(x)}_{\Delta g(x)}\Rightarrow$$

$$\Rightarrow\Delta y=\Delta f(x)+\Delta g(x)\Rightarrow\Delta y=\Delta f(x)+\Delta g(x)\ /\cdot\frac{1}{\Delta x}\Rightarrow\frac{\Delta y}{\Delta x}=\frac{\Delta f(x)}{\Delta x}+\frac{\Delta g(x)}{\Delta x}\Rightarrow$$

$$\Rightarrow\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left(\frac{\Delta f(x)}{\Delta x}+\frac{\Delta g(x)}{\Delta x}\right)\Rightarrow\left[\begin{array}{l}\text{limes zbroja jednak}\\ \text{je zbroju limesa}\end{array}\right]\Rightarrow$$

$$\Rightarrow\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta f(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{\Delta g(x)}{\Delta x}\Rightarrow\left[\begin{array}{l}\text{po definiciji}\\ \text{derivacije}\end{array}\right]\Rightarrow$$

$$\Rightarrow y'(x)=f'(x)+g'(x)\Rightarrow(f+g)'=f'+g'.$$

Vježba 021

Dokažite poučak o derivaciji razlike: (f – g) ' = f ' – g '.

Rezultat: Dokaz analogan.

Zadatak 022 (Marinela, gimnazija)

Dokažite da je derivacija nezavisne varijable x jednaka jedinici.

Rješenje 022

Stavimo da je

$$y(x)=x.$$

Tada je

$$y(x+\Delta x)=x+\Delta x.$$

Prirast funkcije iznosi:

$$\Delta y=y(x+\Delta x)-y(x)\Rightarrow\Delta y=x+\Delta x-x\Rightarrow\Delta y=\Delta x\Rightarrow\Delta y=\Delta x\ /\cdot\frac{1}{\Delta x}\Rightarrow\frac{\Delta y}{\Delta x}=\frac{\Delta x}{\Delta x}\Rightarrow$$

$$\Rightarrow\frac{\Delta y}{\Delta x}=1\Rightarrow\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=1\Rightarrow x'=1.$$

Vježba 022

Dokažite da je derivacija konstante jednaka nuli.

Rezultat: Stavimo da je $y(x)=c.$ Tada je $y(x+\Delta x)=c.$ Prirast funkcije iznosi:

$$\Delta y=y(x+\Delta x)-y(x)\Rightarrow\Delta y=c-c\Rightarrow\Delta y=0\Rightarrow\Delta y=0\ /\cdot\frac{1}{\Delta x}\Rightarrow\frac{\Delta y}{\Delta x}=0\Rightarrow$$

$$\Rightarrow\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=0\Rightarrow c'=0.$$

Zadatak 023 (Sanja, informatika)

Nađite derivaciju funkcije: $f(x)=\frac{\sin x\cdot tg\,x}{ctg\,x}$.

Rješenje 023

Ponovimo!

$$tg\,x=\frac{\sin x}{\cos x}\quad,\quad ctg\,x=\frac{\cos x}{\sin x}\quad,\quad \cos^2 x+\sin^2 x=1\quad,\quad \left(a^n\right)^m=a^{n\cdot m}$$

Derivacija kvocijenta: $\left(\frac{f(x)}{g(x)}\right)'=\frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{\left(g(x)\right)^2},\ g(x)\neq 0$

Prije deriviranja transformirat ćemo zadanu funkciju:

$$f(x)=\frac{\sin x\cdot tg\,x}{ctg\,x}\Rightarrow f(x)=\frac{\sin x\cdot\frac{\sin x}{\cos x}}{\frac{\cos x}{\sin x}}\Rightarrow f(x)=\frac{\sin^3 x}{\cos^2 x}.$$

Sada deriviramo po pravilu za derivaciju kvocijenta:

$$f'(x)=\left(\frac{\sin^3 x}{\cos^2 x}\right)'=\frac{\left(\sin^3 x\right)'\cdot\cos^2 x-\sin^3 x\cdot\left(\cos^2 x\right)'}{\left(\cos^2 x\right)^2}=$$

$$=\frac{3\cdot\sin^2 x\cdot(\sin x)'\cdot\cos^2 x-\sin^3 x\cdot 2\cdot\cos x\cdot(\cos x)'}{\cos^4 x}=$$

$$=\frac{3\cdot\sin^2 x\cdot\cos x\cdot\cos^2 x-\sin^3 x\cdot 2\cdot\cos x\cdot(-\sin x)}{\cos^4 x}=\frac{3\cdot\sin^2 x\cdot\cos^3 x+2\cdot\sin^4 x\cdot\cos x}{\cos^4 x}=$$

$$=\frac{\sin^2 x\cdot\cos x\cdot\left(3\cdot\cos^2 x+2\cdot\sin^2 x\right)}{\cos^4 x}=\frac{\sin^2 x\cdot\cos x\cdot\left(3\cdot\cos^2 x+2\cdot\sin^2 x\right)}{\cos^4 x}=$$

$$=\frac{\sin^2 x\cdot\left(3\cdot\cos^2 x+2\cdot\sin^2 x\right)}{\cos^3 x}=\frac{\sin^2 x\cdot\left(3\cdot\cos^2 x+2\cdot\left(1-\cos^2 x\right)\right)}{\cos^3 x}=$$

$$=\frac{\sin^2 x\cdot\left(3\cdot\cos^2 x+2-2\cdot\cos^2 x\right)}{\cos^3 x}=\frac{\sin^2 x\cdot\left(2+\cos^2 x\right)}{\cos^3 x}.$$

Vježba 023

Nađite derivaciju funkcije: $f(x)=\frac{\cos x\cdot ctg\,x}{tg\,x}$.

Rezultat: $f'(x)=\frac{-\cos^2 x\cdot\left(2+\sin^2 x\right)}{\sin^3 x}$.

Zadatak 024 (Sanja, informatika)

Nađite derivaciju funkcije: $\ln y+arctg\frac{x}{y}=x\cdot y$.

Rješenje 024

Ponovimo!

$$(\ln y)' = \frac{y'}{y} \text{ jer je } y \text{ funkcija od } x.$$

Ako je zavisnost između x i derivabilne funkcije y zadana u implicitnom obliku F(x, y) = 0, onda je za računanje derivacije y ' u jednostavnijim slučajevima dovoljno:

- izračunati derivaciju po x lijeve strane jednadžbe F(x, y) = 0, smatrajući y funkcijom od x
- izjednačiti tu derivaciju s nulom, tj. staviti da je $\frac{d}{dx}F(x, y) = 0$
- riješiti jednadžbu po y '.

Računamo derivaciju funkcije:

$$\ln y + arctg\frac{x}{y} = x \cdot y \Rightarrow \ln y + arctg\frac{x}{y} - x \cdot y = 0 \Rightarrow \begin{bmatrix} \text{deriviramo lijevu} \\ \text{stranu jednadžbe} \end{bmatrix} \Rightarrow$$

$$\Rightarrow \left(\ln y + arctg\frac{x}{y} - x \cdot y\right)' = 0 \Rightarrow (\ln y)' + \left(arctg\frac{x}{y}\right)' - (x \cdot y)' = 0 \Rightarrow$$

$$\Rightarrow \frac{y'}{y} + \frac{1}{1 + \frac{x^2}{y^2}} \cdot \left(\frac{x}{y}\right)' - \left(x' \cdot y + x \cdot y'\right) = 0 \Rightarrow \frac{y'}{y} + \frac{1}{1 + \frac{x^2}{y^2}} \cdot \frac{x' \cdot y - x \cdot y'}{y^2} - \left(x' \cdot y + x \cdot y'\right) = 0 \Rightarrow$$

$$\Rightarrow \frac{y'}{y} + \frac{1}{\frac{y^2 + x^2}{y^2}} \cdot \frac{1 \cdot y - x \cdot y'}{y^2} - \left(1 \cdot y + x \cdot y'\right) = 0 \Rightarrow \frac{y'}{y} + \frac{y^2}{y^2 + x^2} \cdot \frac{y - x \cdot y'}{y^2} - \left(y + x \cdot y'\right) = 0 \Rightarrow$$

$$\Rightarrow \frac{y'}{y} + \frac{y^2}{y^2 + x^2} \cdot \frac{y - x \cdot y'}{y^2} - \left(y + x \cdot y'\right) = 0 \Rightarrow \frac{y'}{y} + \frac{y - x \cdot y'}{y^2 + x^2} - y - x \cdot y' = 0 \Rightarrow \begin{bmatrix} \text{množimo} \\ \text{jednadžbu} \\ \text{zajedničkim} \\ \text{nazivnikom} \\ y \cdot \left(y^2 + x^2\right) \end{bmatrix} \Rightarrow$$

$$\Rightarrow \frac{y'}{y} + \frac{y - x \cdot y'}{y^2 + x^2} - y - x \cdot y' = 0 \; / \cdot y \cdot \left(y^2 + x^2\right) \Rightarrow$$

$$\Rightarrow y' \cdot \left(y^2 + x^2\right) + y \cdot \left(y - x \cdot y'\right) - y^2 \cdot \left(y^2 + x^2\right) - x \cdot y' \cdot y \cdot \left(y^2 + x^2\right) = 0 \Rightarrow$$

$$\Rightarrow y' \cdot \left(y^2 + x^2\right) + y^2 - x \cdot y \cdot y' - y^4 - x^2 \cdot y^2 - x \cdot y^3 \cdot y' - x^3 \cdot y \cdot y' = 0 \Rightarrow$$

$$\Rightarrow \begin{bmatrix} \text{na lijevoj strani jednadžbe ostavimo sve članove} \\ \text{koji sadrže } y' \text{, a ostale prebacimo na desnu stranu} \end{bmatrix} \Rightarrow$$

$$\Rightarrow y' \cdot \left(y^2 + x^2\right) - x \cdot y \cdot y' - x \cdot y^3 \cdot y' - x^3 \cdot y \cdot y' = -y^2 + y^4 + x^2 \cdot y^2 \Rightarrow \begin{bmatrix} \text{na lijevoj strani} \\ \text{jednadžbe izlučimo } y' \end{bmatrix} \Rightarrow$$

$$\Rightarrow y' \cdot \left(y^2 + x^2 - x \cdot y - x \cdot y^3 - x^3 \cdot y\right) = y^4 + x^2 \cdot y^2 - y^2 \Rightarrow y' = \frac{y^4 + x^2 \cdot y^2 - y^2}{y^2 + x^2 - x \cdot y - x \cdot y^3 - x^3 \cdot y} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} \text{u brojniku izlučimo } y^2 \\ \text{u nazivniku grupiramo} \end{bmatrix} \Rightarrow y' = \frac{y^2 \cdot \left(y^2 + x^2 - 1\right)}{\left(y^2 + x^2\right) + \left(-x \cdot y^3 - x^3 \cdot y\right) - x \cdot y} \Rightarrow \begin{bmatrix} \text{u nazivniku iz} \\ \text{druge zagrade} \\ \text{izlučimo } - x \cdot y \end{bmatrix} \Rightarrow$$

$$\Rightarrow y' = \frac{y^2 \cdot \left(y^2 + x^2 - 1\right)}{\left(y^2 + x^2\right) - x \cdot y \cdot \left(y^2 + x^2\right) - x \cdot y} \Rightarrow \begin{bmatrix} \text{u nazivniku iz prva dva} \\ \text{člana izlučimo zagradu} \\ \left(y^2 + x^2\right) \end{bmatrix} \Rightarrow$$

$$\Rightarrow y' = \frac{y^2 \cdot \left(y^2 + x^2 - 1\right)}{\left(y^2 + x^2\right) \cdot (1 - x \cdot y) - x \cdot y}.$$

Vježba 024

Nađite derivaciju funkcije: $\ln y - x = 0$.

Rezultat: y ' = y.

Zadatak 025 (Sanja, informatika)

Nađite derivaciju funkcije: $f(x) = x^2 \cdot \cos^2 2x^2$.

Rješenje 025

Ponovimo!

Derivacija zbroja i razlike: $(f \pm g)' = f' \pm g'$.

Derivacija umnoška: $(f \cdot g)' = f' \cdot g + f \cdot g'$.

Sinus dvostrukog kuta: $2 \cdot \sin x \cdot \cos x = \sin 2x$.

Računamo prvu derivaciju:

$$f'(x) = \left(x^2 \cdot \cos^2 2x^2\right)' = \left(x^2\right)' \cdot \cos^2 2x^2 + x^2 \cdot \left(\cos^2 2x^2\right)' =$$

$$= 2 \cdot x \cdot \cos^2 2x^2 + x^2 \cdot 2 \cdot \cos 2x^2 \cdot \left(-\sin 2x^2\right) \cdot 4x = 2 \cdot x \cdot \cos^2 2x^2 - 8 \cdot x^3 \cdot \cos 2x^2 \cdot \sin 2x^2 =$$

$$= 2 \cdot x \cdot \cos^2 2x^2 - 4 \cdot x^3 \cdot \underbrace{2 \cdot \cos 2x^2 \cdot \sin 2x^2}_{\sin 4x^2} = 2 \cdot x \cdot \cos^2 2x^2 - 4 \cdot x^3 \cdot \sin 4x^2.$$

Sada tražimo drugu derivaciju:

$$f''(x) = \left(f'(x)\right)' = \left(2 \cdot x \cdot \cos^2 2x^2 - 4 \cdot x^3 \cdot \sin 4x^2\right)' = \left(2 \cdot x \cdot \cos^2 2x^2\right)' - \left(4 \cdot x^3 \cdot \sin 4x^2\right)' =$$

$$= \left((2 \cdot x)' \cdot \cos^2 2x^2 + 2 \cdot x \cdot \left(\cos^2 2x^2\right)'\right) - \left(\left(4 \cdot x^3\right)' \cdot \sin 4x^2 + 4 \cdot x^3 \cdot \left(\sin 4x^2\right)'\right) =$$

$$= (2 \cdot x)' \cdot \cos^2 2x^2 + 2 \cdot x \cdot \left(\cos^2 2x^2\right)' - \left(4 \cdot x^3\right)' \cdot \sin 4x^2 - 4 \cdot x^3 \cdot \left(\sin 4x^2\right)' =$$

$$= 2 \cdot \cos^2 2x^2 + 2 \cdot x \cdot 2 \cdot \cos 2x^2 \cdot \left(-\sin 2x^2\right) \cdot 4 \cdot x - 12 \cdot x^2 \cdot \sin 4x^2 - 4 \cdot x^3 \cdot \cos 4x^2 \cdot 8 \cdot x =$$

$$= 2 \cdot \cos^2 2x^2 - 16 \cdot x^2 \cdot \cos 2x^2 \cdot \sin 2x^2 - 12 \cdot x^2 \cdot \sin 4x^2 - 32 \cdot x^4 \cdot \cos 4x^2 =$$

$$= 2 \cdot \cos^2 2x^2 - 8 \cdot x^2 \cdot 2 \cdot \cos 2x^2 \cdot \sin 2x^2 - 12 \cdot x^2 \cdot \sin 4x^2 - 32 \cdot x^4 \cdot \cos 4x^2 =$$

$$= 2\cdot\cos^2 2x^2 - 8\cdot x^2\cdot\underbrace{2\cdot\cos 2x^2\cdot\sin 2x^2}_{\sin 4x^2} - 12\cdot x^2\cdot\sin 4x^2 - 32\cdot x^4\cdot\cos 4x^2 =$$

$$= 2\cdot\cos^2 2x^2 - 8\cdot x^2\cdot\sin 4x^2 - 12\cdot x^2\cdot\sin 4x^2 - 32\cdot x^4\cdot\cos 4x^2 =$$

$$= 2\cdot\cos^2 2x^2 - 20\cdot x^2\cdot\sin 4x^2 - 32\cdot x^4\cdot\cos 4x^2.$$

Vježba 025

Nađite derivaciju funkcije: $f(x) = x\cdot\sin x.$

Rezultat: $f''(x) = 2\cdot\cos x - x\cdot\sin x.$

Zadatak 026 (Sanja, informatika)

Odredite derivaciju y ' iz jednadžbe $2\cdot x + 100\cdot\sqrt{e} = 1 + x\cdot y^3$ u točki T(1, 1).

Rješenje 026

Ponovimo!

$$\left(y^n\right)' = n\cdot y^{n-1}\cdot y' \text{ jer je } y \text{ funkcija od } x.$$

Derivacija zbroja i razlike: $(f \pm g)' = f' \pm g'.$

Derivacija umnoška: $(f\cdot g)' = f'\cdot g + f\cdot g'.$

Derivacija konstante: $c' = 0$, $100' = 0$, $\left(100\cdot\sqrt{e}\right)' = 0.$

Ako je zavisnost između x i derivabilne funkcije y zadana u implicitnom obliku F(x, y) = 0, onda je za računanje derivacije y ' u jednostavnijim slučajevima dovoljno:

- izračunati derivaciju po x lijeve strane jednadžbe F(x, y) = 0, smatrajući y funkcijom od x
- izjednačiti tu derivaciju s nulom, tj. staviti da je $\frac{d}{dx}F(x, y) = 0$
- riješiti jednadžbu po y '.

Računamo derivaciju funkcije:

$$2\cdot x + 100\cdot\sqrt{e} = 1 + x\cdot y^3 \Rightarrow 2\cdot x + 100\cdot\sqrt{e} - 1 - x\cdot y^3 = 0 \Rightarrow \begin{bmatrix}\text{deriviramo lijevu}\\ \text{stranu jednadžbe}\end{bmatrix} \Rightarrow$$

$$\Rightarrow \left(2\cdot x + 100\cdot\sqrt{e} - 1 - x\cdot y^3\right)' = 0 \Rightarrow (2\cdot x)' + \left(100\cdot\sqrt{e}\right)' - 1' - \left(x\cdot y^3\right)' = 0 \Rightarrow$$

$$\Rightarrow 2\cdot 1 + 0 - 0 - \left(1\cdot y^3 + x\cdot 3\cdot y^2\cdot y'\right) = 0 \Rightarrow 2 - y^3 - 3\cdot x\cdot y^2\cdot y' = 0 \Rightarrow \begin{bmatrix}\text{rješavamo}\\ \text{jednadžbu po } y'\end{bmatrix} \Rightarrow$$

$$\Rightarrow -3\cdot x\cdot y^2\cdot y' = -2 + y^3 \;/\cdot(-1) \Rightarrow 3\cdot x\cdot y^2\cdot y' = 2 - y^3 \Rightarrow y' = \frac{2 - y^3}{3\cdot x\cdot y^2}.$$

Koordinate točke T uvrstimu u dobivenu derivaciju:

$$\left.\begin{array}{l} T(x, y) = T(1, 1) \\ y' = \dfrac{2 - y^3}{3\cdot x\cdot y^2} \end{array}\right\} \Rightarrow y'_T = \frac{2 - 1^3}{3\cdot 1\cdot 1^2} = \frac{2 - 1}{3\cdot 1\cdot 1} = \frac{1}{3}.$$

Vježba 026

Odredite derivaciju y ' iz jednadžbe $2\cdot x + 100\cdot\sqrt{e} = 1 + x\cdot y^3$ u točki T(2, 1).

Rezultat: $\frac{1}{6}.$

Zadatak 027 (Sanja, informatika)

Dokažite da je derivacija od e^x jednaka e^x.

Rješenje 027

Ponovimo!

Oznake za derivaciju su:

$$y' = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x).$$

Važni limesi: $\lim_{x \to 0} \frac{e^x - 1}{x} = 1 \quad , \quad \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e.$

Logaritam potencije: $\log a^n = n \cdot \log a \quad , \quad \ln a^n = n \cdot \ln a.$

1.inačica

Zadana je funkcija f(x) = e^x. Uzmimo prirast Δx varijable x. Tada prirast Δy funkcije y = f(x) iznosi:

$$\Delta y = f(x+\Delta x) - f(x) \Rightarrow \Delta y = e^{x+\Delta x} - e^x \Rightarrow \Delta y = e^x \cdot \left(e^{\Delta x} - 1\right).$$

Računamo derivaciju y ' :

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \Rightarrow y' = \lim_{\Delta x \to 0} \frac{e^x \cdot \left(e^{\Delta x} - 1\right)}{\Delta x} \Rightarrow y' = \lim_{\Delta x \to 0} e^x \cdot \frac{e^{\Delta x} - 1}{\Delta x} \Rightarrow$$

$$\Rightarrow y' = e^x \cdot \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} \Rightarrow y' = e^x \cdot 1 \Rightarrow y' = e^x.$$

Prema tome je $\left(e^x\right)' = e^x.$

2.inačica

Zadana je funkcija f(x) = e^x. Uzmimo prirast Δx varijable x. Tada prirast Δy funkcije y = f(x) iznosi:

$$\Delta y = f(x+\Delta x) - f(x) \Rightarrow \Delta y = e^{x+\Delta x} - e^x \Rightarrow \Delta y = e^x \cdot \left(e^{\Delta x} - 1\right).$$

Postavimo kvocijent:

$$\frac{\Delta y}{\Delta x} = \frac{e^x \cdot \left(e^{\Delta x} - 1\right)}{\Delta x} \Rightarrow \frac{\Delta y}{\Delta x} = e^x \cdot \frac{e^{\Delta x} - 1}{\Delta x}.$$

Ako stavimo zamjenu $t = e^{\Delta x} - 1,$ slijedi:

$$\left.\begin{array}{l} t = e^{\Delta x} - 1 \\ \Delta x \to 0 \Rightarrow t \to 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} e^{\Delta x} = 1 + t \\ \Delta x \to 0 \Rightarrow t \to 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} e^{\Delta x} = 1 + t \ /\ln \\ \Delta x \to 0 \Rightarrow t \to 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \ln e^{\Delta x} = \ln(1+t) \\ \Delta x \to 0 \Rightarrow t \to 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} \Delta x \cdot \ln e = \ln(1+t) \\ \Delta x \to 0 \Rightarrow t \to 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \Delta x \cdot 1 = \ln(1+t) \\ \Delta x \to 0 \Rightarrow t \to 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \Delta x = \ln(1+t) \\ \Delta x \to 0 \Rightarrow t \to 0 \end{array}\right\} \Rightarrow y' = \lim_{x \to 0} \frac{\Delta y}{\Delta x} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} t = e^{\Delta x} - 1 \ , \ \Delta x = \ln(1+t) \ , \ \dfrac{\Delta y}{\Delta x} = \dfrac{e^x \cdot \left(e^{\Delta x} - 1\right)}{\Delta x} \\ \Delta x \to 0 \Rightarrow t \to 0 \end{array}\right\} \Rightarrow y' = \lim_{t \to 0} e^x \cdot \frac{t}{\ln(1+t)} \Rightarrow$$

$$\Rightarrow y' = e^x \cdot \lim_{t\to 0} \frac{t}{\ln(1+t)} \Rightarrow y' = e^x \cdot \lim_{t\to 0} \frac{1}{\frac{\ln(1+t)}{t}} \Rightarrow y' = e^x \cdot \lim_{t\to 0} \frac{1}{\frac{1}{t}\cdot \ln(1+t)} \Rightarrow$$

$$\Rightarrow y' = e^x \cdot \lim_{t\to 0} \frac{1}{\ln(1+t)^{\frac{1}{t}}} \Rightarrow y' = e^x \cdot \frac{1}{\lim_{t\to 0} \ln(1+t)^{\frac{1}{t}}} \Rightarrow \left[\begin{array}{l}\text{Funkcija } y=\ln x \text{ je neprekidna,}\\ \text{znači da je } \ln a = \lim_{x\to a} \ln x,\\ \ln \text{ i } \lim \text{ mogu zamijeniti mjesta}\end{array}\right] \Rightarrow$$

$$\Rightarrow y' = e^x \cdot \frac{1}{\ln\left(\lim_{t\to 0} (1+t)^{\frac{1}{t}}\right)} \Rightarrow y' = e^x \cdot \frac{1}{\ln e} \Rightarrow y' = e^x \cdot \frac{1}{1} \Rightarrow y' = e^x.$$

Prema tome je $\left(e^x\right)' = e^x$.

Vježba 027

Dokažite da je derivacija od n · e^x jednaka n · e^x.

Rezultat: Izvod analogan.

Zadatak 028 (Sanja, informatika)

Derivirajte funkciju: $y = x^{\sin^x x}$.

Rješenje 028

Ponovimo!

Logaritam potencije: $\log a^n = n \cdot \log a$, $\ln a^n = n \cdot \ln a$.

Logaritam produkta: $\log(a \cdot b) = \log a + \log b$, $\ln(a \cdot b) = \ln a + \ln b$.

Kotangens: $ctg\, x = \frac{\cos x}{\sin x}$.

Logaritamskom derivacijom funkcije y = f(x) nazivamo derivaciju logaritma te funkcije, tj.

$$(\ln y)' = \frac{1}{y} \cdot y' \quad \text{jer je y funkcija od } x.$$

$$y = x^{\sin^x x} \Rightarrow \left[\begin{array}{l}\text{logaritmiramo}\\ \text{funkciju}\end{array}\right] \Rightarrow y = x^{\sin^x x} \;/\ln \Rightarrow \ln y = \ln x^{\sin^x x} \Rightarrow \ln y = \sin^x x \cdot \ln x \Rightarrow$$

$$\Rightarrow \left[\begin{array}{l}\text{ponovno}\\ \text{logaritmiramo}\\ \text{funkciju}\end{array}\right] \Rightarrow \ln y = \sin^x x \cdot \ln x \;/\ln \Rightarrow \ln\ln y = \ln\left(\sin^x x \cdot \ln x\right) \Rightarrow$$

$$\Rightarrow \ln\ln y = \ln\left(\sin^x x\right) + \ln(\ln x) \Rightarrow \ln\ln y = x \cdot \ln(\sin x) + \ln(\ln x) \Rightarrow \left[\begin{array}{l}\text{deriviramo}\\ \text{jednadžbu}\end{array}\right] \Rightarrow$$

$$\Rightarrow (\ln\ln y)' = \left(x \cdot \ln(\sin x) + \ln(\ln x)\right)' \Rightarrow \frac{1}{\ln y} \cdot \frac{1}{y} \cdot y' = \left(x \cdot \ln(\sin x)\right)' + \left(\ln(\ln x)\right)' \Rightarrow$$

$$\Rightarrow \frac{y'}{y \cdot \ln y} = 1 \cdot \ln(\sin x) + x \cdot \frac{1}{\sin x} \cdot \cos x + \frac{1}{\ln x} \cdot \frac{1}{x} \Rightarrow \frac{y'}{y \cdot \ln y} = \ln(\sin x) + x \cdot ctg\, x + \frac{1}{x \cdot \ln x} \Rightarrow$$

$$\Rightarrow \frac{y'}{y \cdot \ln y} = \ln(\sin x) + x \cdot ctg\, x + \frac{1}{x \cdot \ln x} \;/\cdot y \cdot \ln y \Rightarrow y' = y \cdot \ln y \cdot \left(\ln(\sin x) + x \cdot ctg\, x + \frac{1}{x \cdot \ln x}\right) \Rightarrow$$

$$\Rightarrow y' = x^{\sin^x x} \cdot \ln x^{\sin^x x} \cdot \left(\ln(\sin x) + x \cdot ctg\, x + \frac{1}{x \cdot \ln x} \right) \Rightarrow$$

$$\Rightarrow y' = x^{\sin^x x} \cdot \sin^x x \cdot \ln x \cdot \left(\ln(\sin x) + x \cdot ctg\, x + \frac{1}{x \cdot \ln x} \right).$$

Vježba 028

Derivirajte funkciju: $y = \sin^x x.$

Rezultat: $y' = \sin^x x \cdot (\ln(\sin x) + x \cdot ctg\, x).$

Zadatak 029 (Sanja, informatika)

Derivirajte funkciju: $y = x^{x^{\sqrt{x}}}.$

Rješenje 029

Ponovimo!

Logaritam potencije: $\log a^n = n \cdot \log a$, $\ln a^n = n \cdot \ln a.$

Logaritam produkta: $\log(a \cdot b) = \log a + \log b$, $\ln(a \cdot b) = \ln a + \ln b.$

Kraćenje: $\frac{\sqrt{x}}{x} = \frac{\sqrt{x}}{(\sqrt{x})^2} = \frac{\sqrt{x}}{\sqrt{x} \cdot \sqrt{x}} = \frac{1}{\sqrt{x}}.$

Logaritamskom derivacijom funkcije y = f(x) nazivamo derivaciju logaritma te funkcije, tj.

$$(\ln y)' = \frac{1}{y} \cdot y' \text{ jer je } y \text{ funkcija od } x.$$

$$y = x^{x^{\sqrt{x}}} \Rightarrow \begin{bmatrix} \text{logaritmiramo} \\ \text{funkciju} \end{bmatrix} \Rightarrow y = x^{x^{\sqrt{x}}} \;/\ln \Rightarrow \ln y = \ln x^{x^{\sqrt{x}}} \Rightarrow \ln y = x^{\sqrt{x}} \cdot \ln x \Rightarrow$$

$$\Rightarrow \begin{bmatrix} \text{ponovno} \\ \text{logaritmiramo} \\ \text{funkciju} \end{bmatrix} \Rightarrow \ln y = x^{\sqrt{x}} \cdot \ln x \Rightarrow /\ln \Rightarrow \ln \ln y = \ln\left(x^{\sqrt{x}} \cdot \ln x \right) \Rightarrow$$

$$\Rightarrow \ln \ln y = \ln\left(x^{\sqrt{x}} \right) + \ln(\ln x) \Rightarrow \ln \ln y = \sqrt{x} \cdot \ln x + \ln(\ln x) \Rightarrow \begin{bmatrix} \text{deriviramo} \\ \text{jednadžbu} \end{bmatrix} \Rightarrow$$

$$\Rightarrow (\ln \ln y)' = \left(\sqrt{x} \cdot \ln x + \ln(\ln x) \right)' \Rightarrow \frac{1}{\ln y} \cdot \frac{1}{y} \cdot y' = \left(\sqrt{x} \cdot \ln x \right)' + \left(\ln(\ln x) \right)' \Rightarrow$$

$$\Rightarrow \frac{1}{\ln y} \cdot \frac{1}{y} \cdot y' = \frac{1}{2 \cdot \sqrt{x}} \cdot \ln x + \sqrt{x} \cdot \frac{1}{x} + \frac{1}{\ln x} \cdot \frac{1}{x} \Rightarrow \frac{y'}{y \cdot \ln y} = \frac{1}{2 \cdot \sqrt{x}} \cdot \ln x + \frac{\sqrt{x}}{x} + \frac{1}{x \cdot \ln x} \Rightarrow$$

$$\Rightarrow \frac{y'}{y \cdot \ln y} = \frac{1}{2 \cdot \sqrt{x}} \cdot \ln x + \frac{1}{\sqrt{x}} + \frac{1}{x \cdot \ln x} \;/\cdot y \cdot \ln y \Rightarrow y' = y \cdot \ln y \cdot \left(\frac{1}{2 \cdot \sqrt{x}} \cdot \ln x + \frac{1}{\sqrt{x}} + \frac{1}{x \cdot \ln x} \right) \Rightarrow$$

$$\Rightarrow y' = x^{x^{\sqrt{x}}} \cdot \ln x^{x^{\sqrt{x}}} \cdot \left(\frac{1}{2 \cdot \sqrt{x}} \cdot \ln x + \frac{1}{\sqrt{x}} + \frac{1}{x \cdot \ln x} \right) \Rightarrow$$

$$\Rightarrow y' = x^{x^{\sqrt{x}}} \cdot x^{\sqrt{x}} \cdot \ln x \cdot \left(\frac{1}{2 \cdot \sqrt{x}} \cdot \ln x + \frac{1}{\sqrt{x}} + \frac{1}{x \cdot \ln x} \right) \Rightarrow$$

$$\Rightarrow y' = x^{x^{\sqrt{x}}} \cdot x^{\sqrt{x}} \cdot \left(\frac{1}{2 \cdot \sqrt{x}} \cdot \ln^2 x + \frac{1}{\sqrt{x}} \cdot \ln x + \frac{1}{x} \right).$$

Vježba 029

Derivirajte funkciju: $y = x^{x^x}$.

Rezultat: $y' = x^{x^x} \cdot x^x \cdot \left(\frac{1}{x} + \ln x + \ln^2 x \right).$

Zadatak 030 (Goga, gimnazija)

Derivirajte funkciju: $y = \arcsin x^2 + \arccos x^2.$

Rješenje 030

Ponovimo!

Derivacija zbroja: $(f+g)' = f' + g'.$

Derivacija funkcije iznosi:

$$y' = \left(\arcsin x^2 + \arccos x^2\right)' \Rightarrow y' = \left(\arcsin x^2\right)' + \left(\arccos x^2\right)' \Rightarrow$$

$$\Rightarrow y' = \frac{1}{\sqrt{1-\left(x^2\right)^2}} \cdot \left(x^2\right)' + \frac{-1}{\sqrt{1-\left(x^2\right)^2}} \cdot \left(x^2\right)' \Rightarrow y' = \frac{1}{\sqrt{1-x^4}} \cdot 2x - \frac{1}{\sqrt{1-x^4}} \cdot 2x \Rightarrow y' = 0.$$

Vježba 030

Derivirajte funkciju: $y = \arcsin x + \arccos x.$

Rezultat: 0.

Zadatak 031 (Goga, gimnazija)

Derivirajte funkciju: $f(t) = \sin t \cdot \sin(t+\varphi).$

Rješenje 031

Ponovimo!

Derivacija umnoška: $(f \cdot g)' = f' \cdot g + f \cdot g'.$

Sinus zbroja: $\sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y.$

Funkciju deriviramo po varijabli t (φ je konstanta):

$$f'(t) = \left(\sin t \cdot \sin(t+\varphi)\right)' \Rightarrow f'(t) = (\sin t)' \cdot \sin(t+\varphi) + \sin t \cdot \left(\sin(t+\varphi)\right)' \Rightarrow$$

$$\Rightarrow f'(t) = \cos t \cdot \sin(t+\varphi) + \sin t \cdot \cos(t+\varphi) \cdot t' \Rightarrow f'(t) = \cos t \cdot \sin(t+\varphi) + \sin t \cdot \cos(t+\varphi) \cdot 1 \Rightarrow$$

$$\Rightarrow f'(t) = \cos t \cdot \sin(t+\varphi) + \sin t \cdot \cos(t+\varphi) \Rightarrow \begin{bmatrix} \text{sinus} \\ \text{zbroja} \end{bmatrix} \Rightarrow f'(t) = \sin(t+t+\varphi) \Rightarrow f'(t) = \sin(2 \cdot t + \varphi).$$

Vježba 031

Derivirajte funkciju: $f(t) = t^2 + \sin t.$

Rezultat: $2 \cdot t + \cos t.$

Zadatak 032 (Leon, ekonomija)

Derivirajte funkciju: $y = |x|.$

Rješenje 032

Ponovimo!

Definicija apsolutne vrijednosti za realne brojeve:

$$|x| = x \text{ za } x \geq 0 \quad , \quad |x| = -x \text{ za } x < 0.$$

Računamo derivaciju:

- $\left.\begin{matrix} y = |x| \\ x > 0 \end{matrix}\right\} \Rightarrow y = x \Rightarrow y' = 1.$
- $\left.\begin{matrix} y = |x| \\ x < 0 \end{matrix}\right\} \Rightarrow y = -x \Rightarrow y' = -1.$

Uočite da y ' (0) ne egzistira.

Vježba 032

Derivirajte funkciju: $y = |2 \cdot x|$.

Rezultat: $y' = 2 \text{ za } x > 0 \quad , \quad y' = -2 \text{ za } x < 0 \quad , \quad y'(0)$ ne egzistira.

Zadatak 033 (Sanja, informatika)

Derivirajte funkciju: $y = \frac{1}{2} \cdot \ln tg \frac{x}{2} - \frac{1}{2} \cdot \frac{\cos x}{\sin^2 x}$.

Rješenje 033

Ponovimo!

Derivacija razlike: $(f - g)' = f' - g'$.

Derivacija umnoška: $(f \cdot g)' = f' \cdot g + f \cdot g'$.

Derivacija kvocijenta: $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$.

Sinus dvostrukog kuta: $2 \cdot \sin x \cdot \cos x = \sin 2x$.

Osnovna trigonometrijska relacija: $\cos^2 x + \sin^2 x = 1$.

Deriviramo funkciju:

$$y = \frac{1}{2} \cdot \ln tg \frac{x}{2} - \frac{1}{2} \cdot \frac{\cos x}{\sin^2 x} \Rightarrow y' = \left(\frac{1}{2} \cdot \ln tg \frac{x}{2} - \frac{1}{2} \cdot \frac{\cos x}{\sin^2 x}\right)' \Rightarrow y' = \left(\frac{1}{2} \cdot \ln tg \frac{x}{2}\right)' - \left(\frac{1}{2} \cdot \frac{\cos x}{\sin^2 x}\right)' \Rightarrow$$

$$\Rightarrow y' = \frac{1}{2} \cdot \left(\ln tg \frac{x}{2}\right)' - \frac{1}{2} \cdot \left(\frac{\cos x}{\sin^2 x}\right)' \Rightarrow y' = \frac{1}{2} \cdot \frac{1}{tg \frac{x}{2}} \cdot \frac{1}{\cos^2 \frac{x}{2}} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{-\sin x \cdot \sin^2 x - \cos x \cdot 2 \cdot \sin x \cdot \cos x}{\left(\sin^2 x\right)^2} \Rightarrow$$

$$\Rightarrow y' = \frac{1}{2} \cdot \frac{1}{\frac{\sin \frac{x}{2}}{\cos \frac{x}{2}}} \cdot \frac{1}{\cos^2 \frac{x}{2}} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{-\sin^3 x - 2 \cdot \sin x \cdot \cos^2 x}{\sin^4 x} \Rightarrow$$

$$\Rightarrow y' = \frac{1}{2 \cdot \sin \frac{x}{2} \cdot \cos \frac{x}{2}} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{\sin x \cdot \left(\sin^2 x + 2 \cdot \cos^2 x\right)}{\sin^4 x} \Rightarrow y' = \frac{1}{2} \cdot \frac{1}{\sin x} + \frac{1}{2} \cdot \frac{\sin^2 x + 2 \cdot \cos^2 x}{\sin^3 x} \Rightarrow$$

$$\Rightarrow y' = \frac{1}{2} \cdot \left(\frac{1}{\sin x} + \frac{\sin^2 x + 2 \cdot \cos^2 x}{\sin^3 x}\right) \Rightarrow y' = \frac{1}{2} \cdot \frac{\sin^2 x + \sin^2 x + 2 \cdot \cos^2 x}{\sin^3 x} \Rightarrow$$

$$\Rightarrow y' = \frac{1}{2} \cdot \frac{2 \cdot \sin^2 x + 2 \cdot \cos^2 x}{\sin^3 x} \Rightarrow y' = \frac{1}{2} \cdot \frac{2 \cdot \left(\sin^2 x + \cos^2 x\right)}{\sin^3 x} \Rightarrow y' = \frac{\sin^2 x + \cos^2 x}{\sin^3 x} \Rightarrow y' = \frac{1}{\sin^3 x}.$$

Vježba 033

Derivirajte funkciju: $y = \ln \sin x.$

Rezultat: ctg x.

Zadatak 034 (Sanja, informatika)

Derivirajte funkciju: $f(x) = \ln b \cdot \log_b x - \ln x \cdot \log x + x \cdot tg\, x.$

Rješenje 034

Ponovimo!

Derivacija zbroja i razlike: $(f \pm g)' = f' \pm g'.$ Derivacija umnoška: $(f \cdot g)' = f' \cdot g + f \cdot g'.$

$$\log_a b = \frac{\log_c a}{\log_c b} \quad , \quad \log_{10} x = \log x \quad , \quad \log_e x = \ln x \quad , \quad \frac{\ln x}{\ln 10} = \frac{\log_e x}{\log_e 10} = \log_{10} x = \log x.$$

Deriviramo funkciju:

$$f'(x) = \left(\ln b \cdot \log_b x - \ln x \cdot \log x + x \cdot tg\, x\right)' = \left(\ln b \cdot \log_b x\right)' - (\ln x \cdot \log x)' + (x \cdot tg\, x)' =$$

$$= \ln b \cdot \left(\log_b x\right)' - \left[(\ln x)' \cdot \log x + \ln x \cdot (\log x)'\right] + \left[x' \cdot tg\, x + x \cdot (tg\, x)'\right] =$$

$$= \ln b \cdot \frac{1}{x \cdot \ln b} - \left[\frac{1}{x} \cdot \log x + \ln x \cdot \frac{1}{x \cdot \ln 10}\right] + \left[1 \cdot tg\, x + x \cdot \frac{1}{\cos^2 x}\right] =$$

$$= \ln b \cdot \frac{1}{x \cdot \ln b} - \left[\frac{1}{x} \cdot \log x + \ln x \cdot \frac{1}{x \cdot \ln 10}\right] + \left[tg\, x + x \cdot \frac{1}{\cos^2 x}\right] = \frac{1}{x} - \frac{\log x}{x} - \frac{\ln x}{x \cdot \ln 10} + tg\, x + \frac{x}{\cos^2 x} =$$

$$= \frac{1}{x} - \frac{\log x}{x} - \frac{\log x}{x} + tg\, x + \frac{x}{\cos^2 x} = \frac{1}{x} - 2 \cdot \frac{\log x}{x} + tg\, x + \frac{x}{\cos^2 x}.$$

Vježba 034

Derivirajte funkciju: $f(x) = \ln x \cdot \log x - \ln a \cdot \log_a x.$

Rezultat: $\frac{2 \cdot \ln x}{x \cdot \ln 10} - \frac{1}{x}.$

Zadatak 035 (Sanja, informatika)

Derivirajte funkciju: $f(x) = \frac{\sqrt{x} \cdot \ln x}{\sin x}.$

Rješenje 035

Ponovimo!

Ako su u = φ(x) i v = ψ(x) funkcije koje imaju derivacije, onda je:

$$\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2} \; (v \neq 0) \quad , \quad (u \cdot v)' = u' \cdot v + u \cdot v'.$$

$$\frac{\sqrt{x}}{x} = \frac{\sqrt{x}}{\left(\sqrt{x}\right)^2} = \frac{1}{\sqrt{x}}.$$

Deriviramo funkciju:

$$f'(x) = \left(\frac{\sqrt{x} \cdot \ln x}{\sin x}\right)' = \frac{\left(\sqrt{x} \cdot \ln x\right)' \cdot \sin x - \left(\sqrt{x} \cdot \ln x\right) \cdot (\sin x)'}{\sin^2 x} =$$

$$=\frac{\left(\left(\sqrt{x}\right)'\cdot\ln x+\sqrt{x}\cdot(\ln x)'\right)\cdot\sin x-\left(\sqrt{x}\cdot\ln x\right)\cdot\cos x}{\sin^2 x}=\frac{\left(\frac{1}{2\cdot\sqrt{x}}\cdot\ln x+\sqrt{x}\cdot\frac{1}{x}\right)\cdot\sin x-\sqrt{x}\cdot\ln x\cdot\cos x}{\sin^2 x}=$$

$$=\frac{\left(\frac{\ln x}{2\cdot\sqrt{x}}+\frac{1}{\sqrt{x}}\right)\cdot\sin x-\sqrt{x}\cdot\ln x\cdot\cos x}{\sin^2 x}=\frac{\frac{1}{2\cdot\sqrt{x}}\cdot(\ln x+2)\cdot\sin x-\sqrt{x}\cdot\ln x\cdot\cos x}{\sin^2 x}=$$

$$=\frac{\frac{\sin x}{2\cdot\sqrt{x}}\cdot(\ln x+2)-\sqrt{x}\cdot\ln x\cdot\cos x}{\sin^2 x}=\left[\begin{array}{l}\text{brojnik i nazivnik}\\ \text{množimo s } 2\cdot\sqrt{x}\end{array}\right]=\frac{\sin x\cdot(\ln x+2)-2\cdot x\cdot\ln x\cdot\cos x}{2\cdot\sqrt{x}\cdot\sin^2 x}.$$

Vježba 035

Derivirajte funkciju: $f(x)=\frac{\ln x}{x}.$

Rezultat: $\frac{1-\ln x}{x^2}.$

Zadatak 036 (Sanja, informatika)

Derivirajte funkciju: $f(x)=\arcsin x^2+\sin^2 x+\sin x^2.$

Rješenje 036

Ponovimo!

Neka su f, g i h derivabilne funkcije na istom intervalu I. Funkcija f + g + h je derivabilna i vrijedi:

(f + g + h) ' = f ' + g ' + h ' (derivacija zbroja).

Sinus dvostrukog kuta: $2\cdot\sin x\cdot\cos x=\sin 2x.$

Deriviramo funkciju:

$$f'(x)=\left(\arcsin x^2+\sin^2 x+\sin x^2\right)'=\left(\arcsin x^2\right)'+\left(\sin^2 x\right)'+\left(\sin x^2\right)'=$$

$$=\frac{1}{\sqrt{1-\left(x^2\right)^2}}\cdot\left(x^2\right)'+2\cdot\sin x\cdot(\sin x)'+\cos x^2\cdot\left(x^2\right)'=\frac{1}{\sqrt{1-x^4}}\cdot 2\cdot x+2\cdot\sin x\cdot\cos x+\cos x^2\cdot 2\cdot x=$$

$$=\frac{2\cdot x}{\sqrt{1-x^4}}+\sin 2x+2\cdot x\cdot\cos x^2.$$

Vježba 036

Derivirajte funkciju: $f(x)=\sin^2 x-\cos^2 x.$

Rezultat: $2\cdot\sin 2x.$

Zadatak 037 (Kiki, studentica)

Derivirajte funkciju: $f(x)=x^2\cdot\sqrt{2\cdot x+12}+\sin\left(x^2+2\cdot x+3\right)+\ln\left(1-x^2\right).$

Rješenje 037

Ponovimo!

Derivacija zbroja i razlike: $(f\pm g)'=f'\pm g'.$ Derivacija umnoška: $(f\cdot g)'=f'\cdot g+f\cdot g'.$

Deriviramo funkciju:

$$f'(x)=\left(x^2\cdot\sqrt{2\cdot x+12}+\sin\left(x^2+2\cdot x+3\right)+\ln\left(1-x^2\right)\right)'=$$

$$=\left(x^2\cdot\sqrt{2\cdot x+12}\right)'+\left(\sin\left(x^2+2\cdot x+3\right)\right)'+\left(\ln\left(1-x^2\right)\right)'=$$

$$=\left(x^2\right)'\cdot\sqrt{2\cdot x+12}+x^2\cdot\left(\sqrt{2\cdot x+12}\right)'+\left(\sin\left(x^2+2\cdot x+3\right)\right)'+\left(\ln\left(1-x^2\right)\right)'=$$

$$=2\cdot x\cdot\sqrt{2\cdot x+12}+x^2\cdot\frac{1}{2\cdot\sqrt{2\cdot x+12}}\cdot 2+\cos\left(x^2+2\cdot x+3\right)\cdot(2\cdot x+2)+\frac{1}{1-x^2}\cdot(-2\cdot x)=$$

$$=2\cdot x\cdot\sqrt{2\cdot x+12}+\frac{x^2}{\sqrt{2\cdot x+12}}+(2\cdot x+2)\cdot\cos\left(x^2+2\cdot x+3\right)-\frac{2\cdot x}{1-x^2}=$$

$$=\frac{2\cdot x\cdot\left(\sqrt{2\cdot x+12}\right)^2+x^2}{\sqrt{2\cdot x+12}}+2\cdot(x+1)\cdot\cos\left(x^2+2\cdot x+3\right)-\frac{2\cdot x}{1-x^2}=$$

$$=\frac{2\cdot x\cdot(2\cdot x+12)+x^2}{\sqrt{2\cdot x+12}}+2\cdot(x+1)\cdot\cos\left(x^2+2\cdot x+3\right)-\frac{2\cdot x}{1-x^2}=$$

$$=\frac{4\cdot x^2+24\cdot x+x^2}{\sqrt{2\cdot x+12}}+2\cdot(x+1)\cdot\cos\left(x^2+2\cdot x+3\right)-\frac{2\cdot x}{1-x^2}=$$

$$=\frac{5\cdot x^2+24\cdot x}{\sqrt{2\cdot x+12}}+2\cdot(x+1)\cdot\cos\left(x^2+2\cdot x+3\right)-\frac{2\cdot x}{1-x^2}=\frac{x\cdot(5\cdot x+24)}{\sqrt{2\cdot x+12}}+2\cdot(x+1)\cdot\cos\left(x^2+2\cdot x+3\right)-\frac{2\cdot x}{1-x^2}.$$

Vježba 037

Derivirajte funkciju: $f(x)=\sin\left(x^2+2\cdot x+5\right)-\ln\left(1-x^2\right).$

Rezultat: $2\cdot(x+1)\cdot\cos\left(x^2+2\cdot x+5\right)+\frac{2\cdot x}{1-x^2}.$

Zadatak 038 (Kiki, studentica)

Nađite jednadžbu tangente krivulje $y=2\cdot\ln x+1$ u točki x = 3. Skicirajte taj graf i tangentu.

Rješenje 038

Budući da točka D pripada krivulji, uvrstit ćemo apscisu x_0 = 3 u jednadžbu krivulje da dobijemo ordinatu točke. Računamo ordinatu točke D(3, y_0) koja leži na grafu funkcije:

$$\left.\begin{array}{l} x_0=3 \\ y_0=f\left(x_0\right) \end{array}\right\}\Rightarrow y_0=f(3)=2\cdot\ln 3+1=\ln 3^2+\ln e=\ln 9+\ln e=\ln 9e\Rightarrow D\left(x_0, y_0\right)=D(3, \ln 9e).$$

Tražena točka ima koordinate $D(3, \ln 9e)$.

Jednadžba tangente u točki D(x_0, y_0) krivulje y = f(x) glasi:

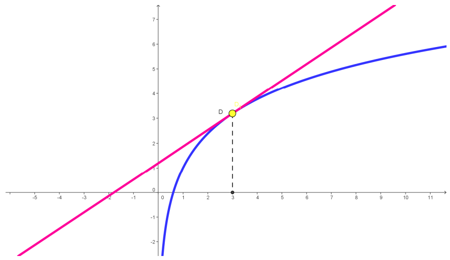
$$y-y_0=f'(x_0)\cdot(x-x_0).$$

Nađemo derivaciju funkcije:

$$f'(x)=(2\cdot\ln x+1)'\Rightarrow f'(x)=2\cdot\frac{1}{x}\Rightarrow f'(x)=\frac{2}{x}$$

i izračunamo njezinu vrijednost za x_0 = 3

$$f'(3)=\frac{2}{3}.$$



Jednadžba tangente je:

$$y - y_0 = f\,'(x_0) \cdot (x - x_0) \Rightarrow$$

$$\Rightarrow y - \ln(9e) = \frac{2}{3} \cdot (x - 3) \Rightarrow y = \frac{2}{3} \cdot x - 2 + \ln(9e) \Rightarrow$$

$$\Rightarrow y = \frac{2}{3} \cdot x - 2 + \ln 9 + \ln e \Rightarrow y = \frac{2}{3} \cdot x - 2 + \ln 9 + 1 \Rightarrow$$

$$\Rightarrow y = \frac{2}{3} \cdot x + \ln 9 - 1.$$

Vježba 038

Nađi jednadžbu tangente krivulje $y = \sqrt{x}$ u točki s apscisom 4.

Rezultat: $y = \frac{1}{4} x + 1.$

Zadatak 039 (Kiki, studentica)

Derivirajte funkciju: $f(x) = \frac{2}{x} + e^x + \ln x + \sqrt{x} + \frac{1}{\sqrt{x}}.$

Rješenje 039

Ponovimo!

Neka su f, g i h derivabilne funkcije na istom intervalu I. Funkcija f + g + h je derivabilna i vrijedi:

(f + g + h) ' = f ' + g ' + h ' (derivacija zbroja).

Računamo derivaciju:

$$f(x) = \frac{2}{x} + e^x + \ln x + \sqrt{x} + \frac{1}{\sqrt{x}} \Rightarrow f(x) = \frac{2}{x} + e^x + \ln x + \sqrt{x} + x^{-\frac{1}{2}} \Rightarrow$$

$$\Rightarrow f\,'(x) = \left(\frac{2}{x} + e^x + \ln x + \sqrt{x} + x^{-\frac{1}{2}}\right)' \Rightarrow f\,'(x) = \left(\frac{2}{x}\right)' + \left(e^x\right)' + (\ln x)' + \left(\sqrt{x}\right)' + \left(x^{-\frac{1}{2}}\right)' \Rightarrow$$

$$\Rightarrow f\,'(x) = -\frac{2}{x^2} + e^x + \frac{1}{x} + \frac{1}{2 \cdot \sqrt{x}} - \frac{1}{2} \cdot x^{-\frac{3}{2}} \Rightarrow f\,'(x) = -\frac{2}{x^2} + e^x + \frac{1}{x} + \frac{1}{2 \cdot \sqrt{x}} - \frac{1}{2 \cdot \sqrt{x^3}} \Rightarrow$$

$$\Rightarrow f\,'(x) = -\frac{2}{x^2} + e^x + \frac{1}{x} + \frac{1}{2 \cdot \sqrt{x}} - \frac{1}{2 \cdot \sqrt{x^2 \cdot x}} \Rightarrow f\,'(x) = -\frac{2}{x^2} + e^x + \frac{1}{x} + \frac{1}{2 \cdot \sqrt{x}} - \frac{1}{2 \cdot x \cdot \sqrt{x}} \Rightarrow$$

$$\Rightarrow f\,'(x) = -\frac{2}{x^2} + e^x + \frac{2 \cdot \sqrt{x} + x - 1}{2 \cdot x \cdot \sqrt{x}} \Rightarrow f\,'(x) = \frac{2 \cdot \sqrt{x} + x - 1}{2 \cdot x \cdot \sqrt{x}} + e^x - \frac{2}{x^2}.$$

Vježba 039

Derivirajte funkciju: $f(x) = \ln x + \frac{1}{x}.$

Rezultat: $\frac{x-1}{x^2}.$

Zadatak 040 (Kiki, studentica)

Derivirajte funkciju: $f(x) = e^x \cdot \sin x + \left(x^2 - 2 \cdot x + 2\right) \cdot \ln x + 2^x \cdot \sqrt{x}.$

Rješenje 040

Ponovimo!

Neka su f, g i h derivabilne funkcije na istom intervalu I. Funkcija f + g + h je derivabilna i vrijedi:

$$(f+g+h)' = f' + g' + h' \text{ (derivacija zbroja)}.$$

Derivacija umnoška: $(f \cdot g)' = f' \cdot g + f \cdot g'$.

Računamo derivaciju:

$$f(x) = e^x \cdot \sin x + \left(x^2 - 2 \cdot x + 2\right) \cdot \ln x + 2^x \cdot \sqrt{x} \Rightarrow f'(x) = \left(e^x \cdot \sin x + \left(x^2 - 2 \cdot x + 2\right) \cdot \ln x + 2^x \cdot \sqrt{x}\right)' \Rightarrow$$

$$\Rightarrow f'(x) = \left(e^x \cdot \sin x\right)' + \left(\left(x^2 - 2 \cdot x + 2\right) \cdot \ln x\right)' + \left(2^x \cdot \sqrt{x}\right)' \Rightarrow$$

$$\Rightarrow f'(x) = \left(e^x\right)' \cdot \sin x + e^x \cdot (\sin x)' + \left(x^2 - 2 \cdot x + 2\right)' \cdot \ln x + \left(x^2 - 2 \cdot x + 2\right) \cdot (\ln x)' + \left(2^x\right)' \cdot \sqrt{x} + 2^x \cdot \left(\sqrt{x}\right)' \Rightarrow$$

$$\Rightarrow f'(x) = e^x \cdot \sin x + e^x \cdot \cos x + (2 \cdot x - 2) \cdot \ln x + \left(x^2 - 2 \cdot x + 2\right) \cdot \frac{1}{x} + 2^x \cdot \ln 2 \cdot \sqrt{x} + 2^x \cdot \frac{1}{2 \cdot \sqrt{x}} \Rightarrow$$

$$\Rightarrow f'(x) = e^x \cdot (\sin x + \cos x) + 2 \cdot (x - 1) \cdot \ln x + \frac{x^2 - 2 \cdot x + 2}{x} + 2^x \cdot \left(\sqrt{x} \cdot \ln 2 + \frac{1}{2 \cdot \sqrt{x}}\right).$$

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Derivirajte funkciju: $f(x) = x^2 \cdot e^x$.

Rezultat: $x \cdot e^x \cdot (x + 2)$.