Zadatak 101 (Ivana, gimnazija)

U Gaussovoj ravnini prikaži skup svih točaka koje su zadane uvjetom:

$$(1-i)\cdot \bar{z}=(1+i)\cdot z\ ,\ z=x+y\cdot i.$$

Rješenje 101

Ponovimo!

$$z=x+y\cdot i\ \Rightarrow\ \bar{z}=x-y\cdot i.$$

$$\left.\begin{array}{l} z=x+y\cdot i\ ,\ \bar{z}=x-y\cdot i \\ (1-i)\cdot \bar{z}=(1+i)\cdot z \end{array}\right\}\Rightarrow\ (1-i)\cdot(x-y\cdot i)=(1+i)\cdot(x+y\cdot i)\ \Rightarrow$$

$$\Rightarrow\ x-y\cdot i-x\cdot i+y\cdot i^2=x+y\cdot i+x\cdot i+y\cdot i^2\ \Rightarrow\ x-y\cdot i-x\cdot i+y\cdot i^2=x+y\cdot i+x\cdot i+y\cdot i^2\ \Rightarrow$$

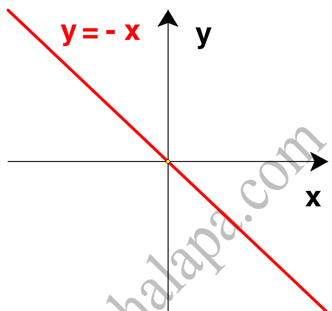
$$\Rightarrow\ -y\cdot i-x\cdot i=y\cdot i+x\cdot i=0\ \Rightarrow\ -y\cdot i-x\cdot i-y\cdot i-x\cdot i=0\ \Rightarrow\ -2\cdot x\cdot i-2\cdot y\cdot i=0\ /:(-2)\ \Rightarrow$$

$$\Rightarrow\ x\cdot i+y\cdot i=0\ \Rightarrow\ (x+y)\cdot i=0.$$

Jednakost (x + y) · i = 0 vrijedi samo ako je

$$x+y=0\ \Rightarrow\ y=-x.$$

Skup svih točaka što je određen danim uvjetom jest pravac, simetrala II. i IV. kvadranta.



Vježba 101

U Gaussovoj ravnini prikaži skup svih točaka koje su zadane uvjetom:

$$(1-i)\cdot z=(1+i)\cdot \bar{z}\ ,\ z=x+y\cdot i.$$

Rezultat: y = x, simetrala I. i III. kvadranta.

Zadatak 102 (Ivana, gimnazija)

Prikaži u Gaussovoj ravnini skup točaka određenih jednadžbom $\left|\frac{z+i}{z-3\cdot i}\right|=1.$

Rješenje 102

Ponovimo!

$$z=x+y\cdot i\ \Rightarrow\ |z|=\sqrt{x^2+y^2}\quad ,\quad \left|\frac{z}{w}\right|=\frac{|z|}{|w|}\quad ,\quad \frac{a}{b}=1\ \Rightarrow\ a=b\quad ,\quad \left(\sqrt{a}\right)^2=a.$$

$$(a+b)^2=a^2+2\cdot a\cdot b+b^2\quad ,\quad (a-b)^2=a^2-2\cdot a\cdot b+b^2.$$

$$\left|\frac{z+i}{z-3\cdot i}\right|=1\ ,\ z\neq 3\cdot i\ \Rightarrow\ \frac{|z+i|}{|z-3\cdot i|}=1\ \Rightarrow\ |z+i|=|z-3\cdot i|.$$

Ako kompleksan broj z prikažemo u standardnom (algebarskom) obliku z = x + y · i, tada je:

$$\left.\begin{array}{l} |z+i|=|z-3\cdot i| \\ z=x+y\cdot i \end{array}\right\}\Rightarrow\ |x+y\cdot i+i|=|x+y\cdot i-3\cdot i|\ \Rightarrow\ |x+(y+1)\cdot i|=|x+(y-3)\cdot i|\ \Rightarrow$$

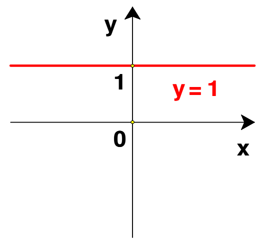
$$\Rightarrow\ \sqrt{x^2+(y+1)^2}=\sqrt{x^2+(y-3)^2}\ \Rightarrow\ \left[\begin{array}{c}\text{kvadriramo}\\ \text{jednadžbu}\end{array}\right]\Rightarrow\ \sqrt{x^2+(y+1)^2}=\sqrt{x^2+(y-3)^2}\ /^2\ \Rightarrow$$

$$\Rightarrow \left(\sqrt{x^2+(y+1)^2}\right)^2=\left(\sqrt{x^2+(y-3)^2}\right)^2 \Rightarrow x^2+(y+1)^2=x^2+(y-3)^2 \Rightarrow$$

$$\Rightarrow x^2+y^2+2\cdot y+1=x^2+y^2-6\cdot y+9 \Rightarrow 2\cdot y+1=-6\cdot y+9 \Rightarrow 2\cdot y+6\cdot y=9-1 \Rightarrow$$

$$\Rightarrow 8\cdot y=8 \;/:8 \Rightarrow y=1.$$

Vidi sliku!



Vježba 102

Prikaži u Gaussovoj ravnini skup točaka određenih jednadžbom $\left|\dfrac{z-3\cdot i}{z+i}\right|=1.$

Rezultat: y = 1.

Zadatak 103 (Tomislav, srednja škola)

Odredite $a, b\in R$ tako da brojevi $z=a-2+(b+3)\cdot i$ i $w=\frac{1}{2}\cdot a+3\cdot b\cdot i$ budu konjugirano kompleksni.

Rješenje 103

Ponovimo!

Neka je z = x + y · i bilo koji kompleksni broj. Sa $\overline{z}$ označavamo broj $\overline{z}=x-y\cdot i$ koji nazivamo kompleksno konjugiranim broju z. Par kompleksno konjugiranih brojeva razlikuju se samo u predznaku imaginarnog dijela:

$$z=x+y\cdot i \Rightarrow \overline{z}=x-y\cdot i.$$

Dva su kompleksna broja jednaka ako i samo ako su im međusobno jednaki realni dijelovi i međusobno jednaki imaginarni dijelovi, tj.

$$a+b\cdot i=c+d\cdot i \Leftrightarrow a=c \;,\; b=d.$$

$$\left.\begin{array}{l} z=a-2+(b+3)\cdot i \\ w=\frac{1}{2}\cdot a+3\cdot b\cdot i \end{array}\right\} \Rightarrow \left[\, w=\overline{z} \,\right] \Rightarrow \left.\begin{array}{l} a-2=\frac{1}{2}\cdot a \\ b+3=-3\cdot b \end{array}\right\} \Rightarrow \left.\begin{array}{l} a-2=\frac{1}{2}\cdot a \;/\cdot 2 \\ b+3\cdot b=-3 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 2\cdot a-4=a \\ 4\cdot b=-3 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} 2\cdot a-a=4 \\ 4\cdot b=-3 \;/:4 \end{array}\right\} \Rightarrow \left.\begin{array}{l} a=4 \\ b=-\frac{3}{4} \end{array}\right\}.$$

Vježba 103

Odredite $a, b\in R$ tako da brojevi $z=a-2+(b+3)\cdot i$ i $w=\frac{1}{3}\cdot a+2\cdot b\cdot i$ budu konjugirano kompleksni.

Rezultat: a = 3 , b = – 1.

Zadatak 104 (Mario, srednja škola)

Ako je $x_1=\dfrac{2+i}{i}$ jedno rješenje kvadratne jednadžbe $x^2+p\cdot x+q=0,\; p, q\in R,$ kako glasi ta jednadžba?

Rješenje 104

Ponovimo!

$$\frac{a+b}{n}=\frac{a}{n}+\frac{b}{n} \quad , \quad i^2=-1 \quad , \quad (a+b\cdot i)\cdot(a-b\cdot i)=a^2+b^2.$$

Neka je z = x + y · i bilo koji kompleksni broj. Sa $\bar{z}$ označavamo broj $\bar{z}=x-y\cdot i$ koji nazivamo kompleksno konjugiranim broju z. Par kompleksno konjugiranih brojeva razlikuju se samo u predznaku imaginarnog dijela:

$$z=x+y\cdot i \;\Rightarrow\; \bar{z}=x-y\cdot i.$$

Ako kvadratna jednadžba ima kompleksna rješenja, tada su to konjugirano kompleksni brojevi:

$$x_1=m+n\cdot i \quad , \quad x_2=m-n\cdot i.$$

Kvadratna jednadžba sa rješenjima x_1 i x_2 glasi

$$a\cdot(x-x_1)\cdot(x-x_2)=0,$$

gdje je $a \neq 0$ po volji odabran broj.

Ako vrijedi

$$x_1+x_2=b \quad , \quad x_1\cdot x_2=c,$$

onda su x_1 i x_2 rješenja kvadratne jednadžbe

$$x^2-b\cdot x+c=0.$$

Najprije nađemo drugo rješenje x_2 kvadratne jednadžbe:

$$x_1=\frac{2+i}{i} \Rightarrow x_1=\frac{2}{i}+\frac{i}{i} \Rightarrow x_1=\frac{2}{i}+1 \Rightarrow x_1=1+\frac{2}{i} \Rightarrow x_1=1+\frac{2}{i}\cdot\frac{-i}{-i} \Rightarrow x_1=1+\frac{-2\cdot i}{-i^2} \Rightarrow$$

$$x_1=1+\frac{-2\cdot i}{-(-1)} \Rightarrow x_1=1+\frac{-2\cdot i}{1} \Rightarrow x_1=1-2\cdot i \Rightarrow \left.\begin{matrix} x_1=1-2\cdot i \\ x_2=1+2\cdot i \end{matrix}\right\}. \text{ rješenja su konjugirano kompleksna}$$

1.inačica

Kvadratna jednadžba glasi:

$$\left.\begin{matrix} x_1=1-2\cdot i \\ x_2=1+2\cdot i \end{matrix}\right\} \Rightarrow \left[(x-x_1)\cdot(x-x_2)=0\right] \Rightarrow (x-(1-2\cdot i))\cdot(x-(1+2\cdot i))=0 \Rightarrow$$

$$\Rightarrow (x-1+2\cdot i)\cdot(x-1-2\cdot i)=0 \Rightarrow x^2-x-2\cdot x\cdot i-x+1+2\cdot i+2\cdot x\cdot i-2\cdot i-4\cdot i^2=0 \Rightarrow$$

$$\Rightarrow x^2-x-2\cdot x\cdot i-x+1+2\cdot i+2\cdot x\cdot i-2\cdot i-4\cdot(-1)=0 \Rightarrow$$

$$\Rightarrow x^2-x-2\cdot x\cdot i-x+1+2\cdot i+2\cdot x\cdot i-2\cdot i+4=0 \Rightarrow x^2-x-2\cdot x\cdot i-x+1+2\cdot i+2\cdot x\cdot i-2\cdot i+4=0 \Rightarrow$$

$$\Rightarrow x^2-2\cdot x+5=0.$$

2.inačica

Kvadratna jednadžba glasi:

$$\left.\begin{matrix} x_1=1-2\cdot i \\ x_2=1+2\cdot i \end{matrix}\right\} \Rightarrow \left[\begin{matrix} x_1+x_2=b \;,\; x_1\cdot x_2=c \\ x^2-b\cdot x+c=0 \end{matrix}\right] \Rightarrow \left.\begin{matrix} x_1+x_2=1-2\cdot i+1+2\cdot i \\ x_1\cdot x_2=(1-2\cdot i)\cdot(1+2\cdot i) \end{matrix}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{matrix} x_1+x_2=1-2\cdot i+1+2\cdot i \\ x_1\cdot x_2=1^2+2^2 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} x_1+x_2=2 \\ x_1\cdot x_2=5 \end{matrix}\right\} \Rightarrow x^2-2\cdot x+5=0.$$

Vježba 104

Ako je $x_1=\frac{i-2}{i}$ jedno rješenje kvadratne jednadžbe $x^2+p\cdot x+q=0,\ p,\ q\in R,$ kako glasi ta jednadžba?

Rezultat: $x^2-2\cdot x+5=0.$

Zadatak 105 (Kiki, maturantica gimnazije)

Izračunaj $\sqrt{z}$, ako je $z=9-40\cdot i.$

Rješenje 105

Ponovimo!

Apsolutna vrijednost kompleksnog broja (modul kompleksnog broja)

$$z=x+y\cdot i\ \Rightarrow\ |z|=\sqrt{x^2+y^2}.$$

$$x=y\ \Rightarrow\ |x|=|y|\quad,\quad \left|\sqrt{z}\right|=\sqrt{|z|}\quad,\quad (x+y\cdot i)^2=x^2+2\cdot x\cdot y\cdot i-y^2.$$

Dva su kompleksna broja jednaka ako i samo ako su im međusobno jednaki realni dijelovi i međusobno jednaki imaginarni dijelovi, tj.

$$a+b\cdot i=c+d\cdot i\ \Leftrightarrow\ a=c\ ,\ b=d.$$

1.inačica

Pretpostavimo da je

$$\sqrt{z}=u+v\cdot i.$$

Tada je

$$\sqrt{9-40\cdot i}=u+v\cdot i\ \Rightarrow\ \begin{bmatrix}\text{kvadriramo}\\ \text{jednakost}\end{bmatrix}\ \Rightarrow\ \sqrt{9-40\cdot i}=u+v\cdot i\ /^2\ \Rightarrow\ \left(\sqrt{9-40\cdot i}\right)^2=(u+v\cdot i)^2\ \Rightarrow$$

$$\Rightarrow\ 9-40\cdot i=u^2+2\cdot u\cdot v\cdot i-v^2\ \Rightarrow\ 9-40\cdot i=u^2-v^2+2\cdot u\cdot v\cdot i\ \Rightarrow\ \begin{bmatrix}\text{jednakost kompleksnih}\\ \text{brojeva}\end{bmatrix}\ \Rightarrow$$

$$\Rightarrow\ \left.\begin{array}{l}u^2-v^2=9\\ 2\cdot u\cdot v=-40\end{array}\right\}\ \Rightarrow\ \left.\begin{array}{l}u^2-v^2=9\\ 2\cdot u\cdot v=-40\ /\cdot\frac{1}{2\cdot u}\end{array}\right\}\ \Rightarrow\ \left.\begin{array}{l}u^2-v^2=9\\ v=-\frac{20}{u}\end{array}\right\}\ \Rightarrow\ u^2-\left(-\frac{20}{u}\right)^2=9\ \Rightarrow$$

$$\Rightarrow\ u^2-\frac{400}{u^2}=9\ \Rightarrow\ u^2-\frac{400}{u^2}=9\ /\cdot u^2\ \Rightarrow\ u^4-400=9\cdot u^2\ \Rightarrow\ u^4-9\cdot u^2-400=0\ \Rightarrow$$

$$\Rightarrow\ \begin{bmatrix}\text{bikvadratna jednadžba,}\\ \text{supstitucija: } t=u^2\end{bmatrix}\Rightarrow\ t^2-9\cdot t-400=0\ \Rightarrow\ \left.\begin{array}{l}t^2-9\cdot t-400=0\\ a=1\ ,\ b=-9\ ,\ c=-400\end{array}\right\}\ \Rightarrow$$

$$\Rightarrow\ \left.\begin{array}{l}a=1\ ,\ b=-9\ ,\ c=-400\\ t_{1,2}=\frac{-b\pm\sqrt{b^2-4\cdot a\cdot c}}{2\cdot a}\end{array}\right\}\ \Rightarrow\ t_{1,2}=\frac{9\pm\sqrt{81-4\cdot 1\cdot(-400)}}{2\cdot 1}\ \Rightarrow\ t_{1,2}=\frac{9\pm\sqrt{81+1600}}{2}\ \Rightarrow$$

$$\Rightarrow\ t_{1,2}=\frac{9\pm\sqrt{1681}}{2}\ \Rightarrow\ t_{1,2}=\frac{9\pm 41}{2}\ \Rightarrow\ \left.\begin{array}{l}t_1=\frac{9+41}{2}\\ t_2=\frac{9-41}{2}\end{array}\right\}\ \Rightarrow\ \left.\begin{array}{l}t_1=\frac{50}{2}\\ t_2=\frac{-32}{2}\end{array}\right\}\ \Rightarrow\ \left.\begin{array}{l}t_1=25\\ t_2=-16\ \text{nema smisla}\end{array}\right\}.$$

Vraćamo se supstituciji:

$$\left.\begin{aligned} t &= 25 \\ t &= u^2 \end{aligned}\right\} \Rightarrow u^2 = 25 \Rightarrow u^2 = 25 \;/\sqrt{} \Rightarrow u_{1,2} = \pm\sqrt{25} \Rightarrow \left.\begin{aligned} u_1 &= 5 \\ u_2 &= -5 \end{aligned}\right\}.$$

Računamo vrijednosti nepoznanice v:

- $$\left.\begin{aligned} u_1 &= 5 \\ v_1 &= -\frac{20}{u_1} \end{aligned}\right\} \Rightarrow v_1 = -\frac{20}{5} \Rightarrow v_1 = -4 \Rightarrow \left.\begin{aligned} u_1 &= 5 \\ v_1 &= -4 \end{aligned}\right\} \Rightarrow \sqrt{9-40\cdot i} = 5-4\cdot i.$$
- $$\left.\begin{aligned} u_2 &= -5 \\ v_2 &= -\frac{20}{u_2} \end{aligned}\right\} \Rightarrow v_2 = -\frac{20}{-5} \Rightarrow v_2 = 4 \Rightarrow \left.\begin{aligned} u_2 &= -5 \\ v_2 &= 4 \end{aligned}\right\} \Rightarrow \sqrt{9-40\cdot i} = -5+4\cdot i.$$

2.inačica

Ako je

$$\sqrt{x+y\cdot i} = u+v\cdot i,$$

tada je

$$u^2 = \frac{1}{2}\cdot\left(|z|+x\right)\ ,\ v^2 = \frac{1}{2}\cdot\left(|z|-x\right),$$

gdje je

$$|z| = \sqrt{x^2+y^2}.$$

Dokaz

- Neka je

$$\sqrt{z} = u+v\cdot i.$$

Za apsolutnu vrijednost ili modul vrijedi:

$$\sqrt{z} = u+v\cdot i \Rightarrow \sqrt{z} = u+v\cdot i \;/\,|\;| \Rightarrow \left|\sqrt{z}\right| = |u+v\cdot i| \Rightarrow \sqrt{|z|} = \sqrt{u^2+v^2} \Rightarrow$$
$$\Rightarrow \sqrt{|z|} = \sqrt{u^2+v^2} \;/^2 \Rightarrow |z| = u^2+v^2 \Rightarrow u^2+v^2 = |z|.$$

- Neka je

$$\sqrt{x+y\cdot i} = u+v\cdot i,$$

tada se kvadriranjem dobije:

$$\sqrt{x+y\cdot i} = u+v\cdot i \Rightarrow \sqrt{x+y\cdot i} = u+v\cdot i \;/^2 \Rightarrow x+y\cdot i = u^2+2\cdot u\cdot v\cdot i - v^2 \Rightarrow$$
$$\Rightarrow x+y\cdot i = u^2-v^2+2\cdot u\cdot v\cdot i.$$

Iz jednakosti kompleksnih brojeva slijedi jednadžba:

$$u^2-v^2 = x.$$

Riješimo sustav jednadžbi po varijablama u^2 i v^2:

$$\left.\begin{aligned} u^2+v^2 &= |z| \\ u^2-v^2 &= x \end{aligned}\right\} \Rightarrow \left[\begin{array}{l}\text{metoda suprotnih} \\ \text{koeficijenata}\end{array}\right] \Rightarrow 2\cdot u^2 = |z|+x \;/:2 \Rightarrow u^2 = \frac{1}{2}\cdot\left(|z|+x\right) \Rightarrow$$

$$\left.\begin{aligned} u^2+v^2 &= |z| \\ u^2 &= \frac{1}{2}\cdot\left(|z|+x\right) \end{aligned}\right\} \Rightarrow \left.\begin{aligned} v^2 &= |z|-u^2 \\ u^2 &= \frac{1}{2}\cdot\left(|z|+x\right) \end{aligned}\right\} \Rightarrow v^2 = |z|-\frac{1}{2}\cdot\left(|z|+x\right) \Rightarrow v^2 = |z|-\frac{1}{2}\cdot|z|-\frac{1}{2}\cdot x \Rightarrow$$

$$\Rightarrow v^2=\frac{1}{2}\cdot|z|-\frac{1}{2}\cdot x \Rightarrow v^2=\frac{1}{2}\cdot(|z|-x).$$

Konačno rješenje je:

$$\left.\begin{array}{l} u^2=\frac{1}{2}\cdot(|z|+x) \\ v^2=\frac{1}{2}\cdot(|z|-x) \end{array}\right\}. \text{ Dokaz gotov.}$$

Računamo $\sqrt{z}$, ako je $z=9-40\cdot i$.

$$z=9-40\cdot i \Rightarrow |z|=\sqrt{9^2+(-40)^2} \Rightarrow |z|=\sqrt{81+1600} \Rightarrow |z|=\sqrt{1681} \Rightarrow |z|=41.$$

$$\left.\begin{array}{l} z=x+y\cdot i \\ z=9-40\cdot i \end{array}\right\} \Rightarrow x=9.$$

- $$\left.\begin{array}{l} u^2=\frac{1}{2}\cdot(|z|+x) \\ |z|=41 \;,\; x=9 \end{array}\right\} \Rightarrow u^2=\frac{1}{2}\cdot(41+9) \Rightarrow u^2=\frac{1}{2}\cdot 50 \Rightarrow u^2=25 \Rightarrow u^2=25 \,/\sqrt{} \Rightarrow$$

$$\Rightarrow u_{1,2}=\pm\sqrt{25} \Rightarrow \left.\begin{array}{l} u_1=5 \\ u_2=-5 \end{array}\right\}.$$

- $$\left.\begin{array}{l} v^2=\frac{1}{2}\cdot(|z|-x) \\ |z|=41 \;,\; x=9 \end{array}\right\} \Rightarrow v^2=\frac{1}{2}\cdot(41-9) \Rightarrow v^2=\frac{1}{2}\cdot 32 \Rightarrow v^2=16 \Rightarrow v^2=16 \,/\sqrt{} \Rightarrow$$

$$\Rightarrow v_{1,2}=\pm\sqrt{16} \Rightarrow \left.\begin{array}{l} v_1=4 \\ v_2=-4 \end{array}\right\}.$$

Rješenja su:

$$\sqrt{9-40\cdot i}=\left.\begin{array}{l} 5-4\cdot i \\ -5+4\cdot i \end{array}\right\}.$$

Vježba 105

Izračunaj $\sqrt{z}$, ako je $z=5-12\cdot i$.

Rezultat: $3-2\cdot i \;,\; -3+2\cdot i.$

Zadatak 106 (Marina, srednja škola)

Riješi jednadžbu: $z^2-3\cdot i\cdot z+10=0.$

Rješenje 106

Ponovimo!

$$i^2=-1.$$

Za svaki pozitivni realni broj p definiramo

$$\sqrt{-p}=i\cdot\sqrt{p}.$$

To je kvadratna jednadžba po nepoznanici z pa vrijedi:

$$z^2-3\cdot i\cdot z+10=0 \Rightarrow \left.\begin{array}{l} z^2-3\cdot i\cdot z+10=0 \\ a=1 \;,\; b=-3\cdot i \;,\; c=10 \end{array}\right\} \Rightarrow \left.\begin{array}{l} a=1 \;,\; b=-3\cdot i \;,\; c=10 \\ z_{1,2}=\frac{-b\pm\sqrt{b^2-4\cdot a\cdot c}}{2\cdot a} \end{array}\right\} \Rightarrow$$

$$\Rightarrow z_{1,2}=\frac{3\cdot i\pm\sqrt{(-3\cdot i)^2-4\cdot 1\cdot 10}}{2\cdot 1} \Rightarrow z_{1,2}=\frac{3\cdot i\pm\sqrt{9\cdot i^2-40}}{2} \Rightarrow z_{1,2}=\frac{3\cdot i\pm\sqrt{-9-40}}{2} \Rightarrow$$

$$\Rightarrow z_{1,2}=\frac{3\cdot i\pm\sqrt{-49}}{2} \Rightarrow z_{1,2}=\frac{3\cdot i\pm 7\cdot i}{2} \Rightarrow \left.\begin{array}{l} z_1=\frac{3\cdot i+7\cdot i}{2} \\ z_2=\frac{3\cdot i-7\cdot i}{2}\end{array}\right\} \Rightarrow \left.\begin{array}{l} z_1=\frac{10\cdot i}{2} \\ z_2=\frac{-4\cdot i}{2}\end{array}\right\} \Rightarrow \left.\begin{array}{l} z_1=5\cdot i \\ z_2=-2\cdot i\end{array}\right\}.$$

Vježba 106

Riješi jednadžbu: $z^2+i\cdot z+6=0.$

Rezultat: $z_1=2\cdot i\ ,\ z_2=-3\cdot i.$

Zadatak 107 (Ana, gimnazija)

Izračunaj x ako je: $z=\frac{\left(x+i\cdot\sqrt{3}\right)^2}{1-i}\ ,\ |z|=2\cdot\sqrt{2}\ ,\ x>0.$

Rješenje 107

Ponovimo!

Apsolutna vrijednost (modul) kompleksnog broja

$$z=x+y\cdot i \Rightarrow |z|=\sqrt{x^2+y^2} \quad , \quad z=x+y\cdot i \Rightarrow |z|^2=x^2+y^2.$$

$$\left|\frac{z}{w}\right|=\frac{|z|}{|w|} \quad , \quad \left|z^n\right|=|z|^n.$$

$$z=\frac{\left(x+i\cdot\sqrt{3}\right)^2}{1-i} \Rightarrow |z|=\left|\frac{\left(x+i\cdot\sqrt{3}\right)^2}{1-i}\right| \Rightarrow |z|=\frac{\left|\left(x+i\cdot\sqrt{3}\right)^2\right|}{|1-i|} \Rightarrow |z|=\frac{\left|x+i\cdot\sqrt{3}\right|^2}{|1-i|} \Rightarrow$$

$$\Rightarrow \left[\,|z|=2\cdot\sqrt{2}\,\right] \Rightarrow 2\cdot\sqrt{2}=\frac{x^2+\left(\sqrt{3}\right)^2}{\sqrt{1^2+(-1)^2}} \Rightarrow 2\cdot\sqrt{2}=\frac{x^2+3}{\sqrt{1+1}} \Rightarrow 2\cdot\sqrt{2}=\frac{x^2+3}{\sqrt{2}}\ /\cdot\sqrt{2} \Rightarrow$$

$$\Rightarrow 2\cdot\left(\sqrt{2}\right)^2=x^2+3 \Rightarrow 2\cdot 2=x^2+3 \Rightarrow 4=x^2+3 \Rightarrow x^2+3=4 \Rightarrow x^2=4-3 \Rightarrow$$

$$\Rightarrow x^2=1\ /\sqrt{\ } \Rightarrow x_{1,2}=\pm\sqrt{1} \Rightarrow \left.\begin{array}{l} x_1=1 \\ x_2=-1 \text{ nije rješenje zbog } x>0\end{array}\right\} \Rightarrow x=1.$$

Vježba 107

Izračunaj x ako je: $z=\frac{\left(x+i\cdot\sqrt{3}\right)^2}{1+i}\ ,\ |z|=2\cdot\sqrt{2}\ ,\ x>0.$

Rezultat: 1.

Zadatak 108 (Nikolina, gimnazija)

Koliko ima prirodnih brojeva n manjih od 2005 za koje je ispunjena jednakost $(1+i)^n=(1-i)^n$ (i je imaginarna jedinica)?

Rješenje 108

Ponovimo!

$$\left(\frac{x}{y}\right)^n=\frac{x^n}{y^n}\quad,\quad (x+y)^2=x^2+2\cdot x\cdot y+y^2\quad,\quad i=\sqrt{-1}\quad,\quad i^2=-1.$$

$$a^{f(x)}=a^{g(x)}\;\Rightarrow\; f(x)=g(x).$$

Potencije imaginarne jedinice i

Za $n\in N$ vrijedi

$$i^{4\cdot n}=1\quad,\quad i^{4\cdot n+1}=i\quad,\quad i^{4\cdot n+2}=-1\quad,\quad i^{4\cdot n+3}=-i.$$

Za kompleksne brojeve oblika

$$z=a+b\cdot i\quad,\quad \bar{z}=a-b\cdot i$$

kažemo da su kompleksno konjugirani jedan drugom. Simbol konjugiranja jest povlaka iznad broja koji se konjugira.

Umnožak kompleksnog broja i njemu konjugiranog broja uvijek je realan broj:

$$\left.\begin{array}{l} z=a+b\cdot i \\ \bar{z}=a-b\cdot i \end{array}\right\}\Rightarrow z\cdot\bar{z}=(a+b\cdot i)\cdot(a-b\cdot i)\;\Rightarrow\; z\cdot\bar{z}=a^2+b^2.$$

Dva kompleksna broja dijelimo tako da dijeljenje najprije zapišemo u obliku razlomka. Taj razlomak zatim proširimo množeći njegov brojnik i nazivnik konjugiranim nazivnikom:

$$\frac{a+b\cdot i}{c+d\cdot i}=\frac{(a+b\cdot i)\cdot(c-d\cdot i)}{(c+d\cdot i)\cdot(c-d\cdot i)}=\frac{(a+b\cdot i)\cdot(c-d\cdot i)}{c^2+d^2}.$$

Opći član a_n aritmetičkog niza s prvim članom a_1 i razlikom d ima oblik

$$a_n=a_1+(n-1)\cdot d\;\Rightarrow\; n=\frac{a_n-a_1}{d}+1.$$

$$(1+i)^n=(1-i)^n\;\Rightarrow\;(1+i)^n=(1-i)^n\;/\cdot\frac{1}{(1-i)^n}\;\Rightarrow\;\frac{(1+i)^n}{(1-i)^n}=1\;\Rightarrow\;\left(\frac{1+i}{1-i}\right)^n=1\;\Rightarrow$$

$$\Rightarrow\left[\begin{array}{l}\text{dijeljenje}\\ \text{kompleksnih}\\ \text{brojeva}\end{array}\right]\Rightarrow\left(\frac{1+i}{1-i}\cdot\frac{1+i}{1+i}\right)^n=1\;\Rightarrow\;\left(\frac{(1+i)^2}{(1-i)\cdot(1+i)}\right)^n=1\;\Rightarrow\;\left(\frac{1+2\cdot i+i^2}{1^2+1^2}\right)^n=1\;\Rightarrow$$

$$\Rightarrow\left(\frac{1+2\cdot i-1}{1+1}\right)^n=1\;\Rightarrow\;\left(\frac{1+2\cdot i-1}{2}\right)^n=1\;\Rightarrow\;\left(\frac{2\cdot i}{2}\right)^n=1\;\Rightarrow\;\left(\frac{2\cdot i}{2}\right)^n=1\;\Rightarrow\; i^n=1.$$

1.inačica

Budući da je samo

$$i^{4\cdot n}=1,$$

(a najveći prirodni broj koji je manji od 2005 i za koji to vrijedi je broj 2004), slijedi:

$$\left.\begin{array}{l} i^{4\cdot n}=1 \\ i^{2004}=1 \end{array}\right\}\Rightarrow 4\cdot n=2004\;/:4\;\Rightarrow\; n=501.$$

2.inačica

Prirodni brojevi manji od 2005 za koje vrijedi dobivena jednakost

$$i^n=1$$

čine slijed (niz):

$$i^4=1\;,\; i^8=1\;,\; i^{12}=1\;,\; i^{16}=1\;,\;\ldots\;,\; i^{2004}=1.$$

Uočimo da eksponenti

$$4, 8, 12, 16, ..., 2004$$

tvore aritmetički niz kojem je prvi član $a_1 = 4$, zadnji član $a_n = 2004$ i razlika $d = 4$.
Broj prirodnih brojeva za koje je ispunjena zadana jednakost iznosi:

$$\left.\begin{array}{l} a_1 = 4 \;\;,\;\; a_n = 2004 \;\;,\;\; d = 4 \\ n = \dfrac{a_n - a_1}{d} + 1 \end{array}\right\} \Rightarrow n = \frac{2004-4}{4} + 1 \Rightarrow n = \frac{2000}{4} + 1 \Rightarrow n = 500 + 1 \Rightarrow n = 501.$$

Vježba 108

Koliko ima prirodnih brojeva n manjih od 1005 za koje je ispunjena jednakost $(1+i)^n = (1-i)^n$ (i je imaginarna jedinica)?

Rezultat: 251.

Zadatak 109 (Mirjana, ekonomska škola)

Nađi $z \in C$ koji zadovoljava jednakost $z \cdot \bar{z} - z - |z|^2 = 2 - i.$

Rješenje 109

Ponovimo!

Za kompleksne brojeve oblika

$$z = a + b \cdot i \quad , \quad \bar{z} = a - b \cdot i$$

kažemo da su kompleksno konjugirani jedan drugom. Simbol konjugiranja jest povlaka iznad broja koji se konjugira.
Umnožak kompleksnog broja i njemu konjugiranog broja uvijek je realan broj:

$$\left.\begin{array}{l} z = a + b \cdot i \\ \bar{z} = a - b \cdot i \end{array}\right\} \Rightarrow z \cdot \bar{z} = (a + b \cdot i) \cdot (a - b \cdot i) \Rightarrow z \cdot \bar{z} = a^2 + b^2.$$

Modul kompleksnog broja z = a + b · i definira se formulom

$$|z| = \sqrt{a^2 + b^2} \quad \text{ili} \quad |z|^2 = z \cdot \bar{z}.$$

$$z \cdot \bar{z} - z - |z|^2 = 2 - i \Rightarrow \left[|z|^2 = z \cdot \bar{z} \right] \Rightarrow |z|^2 - z - |z|^2 = 2 - i \Rightarrow |z|^2 - z - |z|^2 = 2 - i \Rightarrow$$

$$\Rightarrow -z = 2 - i \;/ \cdot (-1) \Rightarrow z = -2 + i.$$

Vježba 109

Nađi $z \in C$ koji zadovoljava jednakost $z \cdot \bar{z} - z - |z|^2 = -2 - i.$

Rezultat: 2 + i.

Zadatak 110 (Denis, ekonomska škola)

Nađi udaljenost točaka koje predočavaju kompleksne brojeve z = – 3 – 4 · i i w = 5 + 2 · i.

Rješenje 110

Ponovimo!

Modul ili apsolutna vrijednost kompleksnog broja z = a + b · i definira se formulom

$$|z| = \sqrt{a^2 + b^2}.$$

Udaljenost točaka koje predočavaju kompleksne brojeve z i w je

$$|z - w|.$$

Udaljenost točaka koje predočavaju kompleksne brojeve z = – 3 – 4 · i i w = 5 + 2 · i iznosi:

$$|z-w| = |-3-4\cdot i-(5+2\cdot i)| = |-3-4\cdot i-5-2\cdot i| = |-8-6\cdot i| = \sqrt{(-8)^2+(-6)^2} =$$
$$= \sqrt{64+36} = \sqrt{100} = 10.$$

Vježba 110

Nađi udaljenost točaka koje predočavaju kompleksne brojeve z = 3 – 4 · i i w = – 5 + 2 · i.

Rezultat: 10.

Zadatak 111 (Martina, srednja škola)

Odredi x tako da broj (x + i) · (3 – 2 · i) bude realan.

Rješenje 111

Ponovimo!

$i^2=-1$, $z=x+y\cdot i \Rightarrow x=\mathrm{Re}(z)$, $y=\mathrm{Im}(z)$, $z=x+y\cdot i$ je realan broj ako je $y=0$.

$$z=(x+i)\cdot(3-2\cdot i)=3\cdot x-2\cdot x\cdot i+3\cdot i-2\cdot i^2=3\cdot x-2\cdot x\cdot i+3\cdot i-2\cdot(-1)=$$
$$=3\cdot x-2\cdot x\cdot i+3\cdot i+2=(3\cdot x+2)+(-2\cdot x+3)\cdot i.$$

Da bi broj

$$z=(3\cdot x+2)+(-2\cdot x+3)\cdot i$$

bio realan mora njegov imaginarni dio biti jednak nuli:

$$-2\cdot x+3=0 \Rightarrow -2\cdot x=-3 \;/:(-2) \Rightarrow x=\frac{3}{2}.$$

Vježba 111

Odredi x tako da broj (x + i) · (3 – 2 · i) bude imaginaran.

Rezultat: $x=-\frac{2}{3}.$

Zadatak 112 (Ivica, gimnazija)

Izračunaj: $(1-i)^4\cdot(1+i)^6.$

Rješenje 112

Ponovimo!

$a^n\cdot a^m=a^{n+m}$, $a^n\cdot b^n=(a\cdot b)^n$, $(a-b\cdot i)\cdot(a+b\cdot i)=a^2+b^2$, $i^2=-1.$

$(a+b)^2=a^2+2\cdot a\cdot b+b^2$, $(a^n)^m=a^{n\cdot m}$, $(a-b)^2=a^2-2\cdot a\cdot b+b^2$, $i^3=-i.$

$$(a+b)^3=a^3+3\cdot a^2\cdot b+3\cdot a\cdot b^2+b^3.$$

1.inačica

$$(1-i)^4\cdot(1+i)^6=(1-i)^4\cdot(1+i)^4\cdot(1+i)^2=((1-i)\cdot(1+i))^4\cdot(1+i)^2=\left(1^2+1^2\right)^4\cdot\left(1+2\cdot i+i^2\right)=$$
$$=(1+1)^4\cdot(1+2\cdot i-1)=2^4\cdot(1+2\cdot i-1)=2^4\cdot 2\cdot i=2^5\cdot i=32\cdot i.$$

2.inačica

$$(1-i)^4\cdot(1+i)^6=\left((1-i)^2\right)^2\cdot\left((1+i)^2\right)^3=\left(1-2\cdot i+i^2\right)^2\cdot\left(1+2\cdot i+i^2\right)^3=$$
$$=(1-2\cdot i-1)^2\cdot(1+2\cdot i-1)^3=(1-2\cdot i-1)^2\cdot(1+2\cdot i-1)^3=(-2\cdot i)^2\cdot(2\cdot i)^3=(-2)^2\cdot i^2\cdot 2^3\cdot i^3=$$
$$=4\cdot(-1)\cdot 8\cdot(-i)=32\cdot i.$$

3.inačica

$$(1-i)^4\cdot(1+i)^6=\left((1-i)^2\right)^2\cdot\left((1+i)^3\right)^2=\left(1-2\cdot i+i^2\right)^2\cdot\left(1^3+3\cdot1^2\cdot i+3\cdot1\cdot i^2+i^3\right)^2=$$

$$=(1-2\cdot i-1)^2\cdot(1+3\cdot i+3\cdot(-1)-i)^2=(1-2\cdot i-1)^2\cdot(1+3\cdot i-3-i)^2=(-2\cdot i)^2\cdot(-2+2\cdot i)^2=$$

$$=(-2)^2\cdot i^2\cdot(2\cdot i-2)^2=4\cdot(-1)\cdot\left((2\cdot i)^2-8\cdot i+4\right)=-4\cdot\left(2^2\cdot i^2-8\cdot i+4\right)=-4\cdot(4\cdot(-1)-8\cdot i+4)=$$

$$=-4\cdot(4\cdot(-1)-8\cdot i+4)=-4\cdot(-4-8\cdot i+4)=-4\cdot(-4-8\cdot i+4)=-4\cdot(-8\cdot i)=32\cdot i.$$

Vježba 112

Izračunaj: $(1-i)^6\cdot(1+i)^6$.

Rezultat: 2^6.

Zadatak 113 (Marija, gimnazija)

Neka su $x_1=-2,\ x_2=1+i\cdot\sqrt{2}$, dva rješenja jednadžbe

$$x^3+b\cdot x^2+c\cdot x+d=0,\ b,\ c,\ d\in R.$$

Nađi $b+c+d$.

Rješenje 113

Ponovimo!

$$(x-y\cdot i)\cdot(x+y\cdot i)=x^2+y^2\quad,\quad\left(\sqrt{x}\right)^2=x.$$

Ako je $x_0=\alpha+\beta\cdot i,\ \alpha,\ \beta\in R,\ \beta\neq0$ korijen (rješenje) jednadžbe

$$a_n\cdot x^n+a_{n-1}\cdot x^{n-1}+a_{n-2}\cdot x^{n-2}+a_{n-3}\cdot x^{n-3}+\ \ldots\ +a_2\cdot x^2+a_1\cdot x^1+a_0=0$$

sa realnim koeficijentima, onda je

$$\overline{x_0}=\alpha-\beta\cdot i$$

također korijen (rješenje) te jednadžbe.

Jednadžba trećeg stupnja

$$a\cdot x^3+b\cdot x^2+c\cdot x+d=0$$

ima točno tri korijena (rješenja).

Neka su $x_1,\ x_2,\ x_3$ korijeni (rješenja) jednadžbe

$$a\cdot x^3+b\cdot x^2+c\cdot x+d=0.$$

Tada vrijede Vièteove formule:

$$x_1+x_2+x_3=-\frac{b}{a}\quad,\quad x_1\cdot x_2+x_1\cdot x_3+x_2\cdot x_3=\frac{c}{a}\quad,\quad x_1\cdot x_2\cdot x_3=-\frac{d}{a}.$$

Neka su $x_1,\ x_2,\ x_3$ korijeni (rješenja) jednadžbe

$$x^3+b\cdot x^2+c\cdot x+d=0.$$

Tada vrijede Vièteove formule:

$$x_1+x_2+x_3=-b\quad,\quad x_1\cdot x_2+x_1\cdot x_3+x_2\cdot x_3=c\quad,\quad x_1\cdot x_2\cdot x_3=-d.$$

Budući da je

$$x_2=1+i\cdot\sqrt{2}$$

rješenje jednadžbe

$$x^3+b\cdot x^2+c\cdot x+d=0,$$

slijedi da je treće rješenje x_3 oblika

$$x_3 = 1 - i \cdot \sqrt{2}.$$

Pomoću Vièteovih formula dobije se zbroj b + c + d:

$$\left.\begin{array}{l} x_1 + x_2 + x_3 = -b \\ x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3 = c \\ x_1 \cdot x_2 \cdot x_3 = -d \end{array}\right\} \Rightarrow \left.\begin{array}{l} -2 + 1 + i \cdot \sqrt{2} + 1 - i \cdot \sqrt{2} = -b \\ -2 \cdot \left(1 + i \cdot \sqrt{2}\right) - 2 \cdot \left(1 - i \cdot \sqrt{2}\right) + \left(1 + i \cdot \sqrt{2}\right) \cdot \left(1 - i \cdot \sqrt{2}\right) = c \\ -2 \cdot \left(1 + i \cdot \sqrt{2}\right) \cdot \left(1 - i \cdot \sqrt{2}\right) = -d \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} -2 + 1 + i \cdot \sqrt{2} + 1 - i \cdot \sqrt{2} = -b \\ \Rightarrow -2 - 2 \cdot i \cdot \sqrt{2} - 2 + 2 \cdot i \cdot \sqrt{2} + 1 + \left(\sqrt{2}\right)^2 = c \\ -2 \cdot \left(1 + \left(\sqrt{2}\right)^2\right) = -d \end{array}\right\} \Rightarrow \left.\begin{array}{l} -2 + 1 + 1 = -b \\ -2 - 2 + 1 + 2 = c \\ -2 \cdot (1 + 2) = -d \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} 0 = -b \\ -1 = c \\ -6 = -d \end{array}\right\} \Rightarrow \left.\begin{array}{l} b = 0 \\ c = -1 \\ d = 6 \end{array}\right\} \Rightarrow b + c + d = 0 - 1 + 6 \Rightarrow b + c + d = 5.$$

Vježba 113

Neka su $x_1 = -2,\ x_2 = 1 - i \cdot \sqrt{2}$, dva rješenja jednadžbe

$$x^3 + b \cdot x^2 + c \cdot x + d = 0,\ b,\ c,\ d \in R.$$

Nađi $b + c + d$.

Rezultat: 5.

Zadatak 114 (Marija, gimnazija)

Koliki je imaginarni dio kompleksnog broja

$$z = \frac{a - b \cdot i}{a + b \cdot i} - \frac{a + b \cdot i}{a - b \cdot i},\ \text{gdje su } a,\ b \in R,\ a \neq 0,\ b \neq 0?$$

Rješenje 114

Ponovimo!

$$z = x + y \cdot i \Rightarrow \operatorname{Re}(z) = x,\ \operatorname{Im}(z) = y \quad , \quad (x - y \cdot i) \cdot (x + y \cdot i) = x^2 + y^2.$$

$$(x + y)^2 = x^2 + 2 \cdot x \cdot y + y^2 \quad , \quad (x - y)^2 = x^2 - 2 \cdot x \cdot y + y^2.$$

$$z = \frac{a - b \cdot i}{a + b \cdot i} - \frac{a + b \cdot i}{a - b \cdot i} \Rightarrow z = \frac{(a - b \cdot i)^2 - (a + b \cdot i)^2}{(a + b \cdot i) \cdot (a - b \cdot i)} \Rightarrow$$

$$\Rightarrow z = \frac{a^2 - 2 \cdot a \cdot b \cdot i + (b \cdot i)^2 - \left(a^2 + 2 \cdot a \cdot b \cdot i + (b \cdot i)^2\right)}{a^2 + b^2} \Rightarrow$$

$$\Rightarrow z = \frac{a^2 - 2 \cdot a \cdot b \cdot i + (b \cdot i)^2 - a^2 - 2 \cdot a \cdot b \cdot i - (b \cdot i)^2}{a^2 + b^2} \Rightarrow$$

$$\Rightarrow z = \frac{a^2 - 2 \cdot a \cdot b \cdot i + (b \cdot i)^2 - a^2 - 2 \cdot a \cdot b \cdot i - (b \cdot i)^2}{a^2 + b^2} \Rightarrow z = \frac{-4 \cdot a \cdot b \cdot i}{a^2 + b^2} \Rightarrow z = -\frac{4 \cdot a \cdot b}{a^2 + b^2} \cdot i.$$

Imaginarni dio zadanog kompleksnog broja iznosi:

$$\operatorname{Im}(z) = -\frac{4 \cdot a \cdot b}{a^2 + b^2}.$$

Vježba 114

Koliki je realni dio kompleksnog broja

$$z = \frac{a - b \cdot i}{a + b \cdot i} - \frac{a + b \cdot i}{a - b \cdot i}, \text{ gdje su } a, b \in R, a \neq 0, b \neq 0?$$

Rezultat: 0.

Zadatak 115 (Željka, srednja škola)

Odredi skup točaka z kompleksne ravnine za koje vrijedi $\operatorname{Re}(z+1) = \operatorname{Im}(z-i)$.

Rješenje 115

Ponovimo!

Kompleksan broj je broj oblika

$$z = x + y \cdot i,$$

gdje su x i y realni brojevi, a i imaginarna jedinica. Broj x zove se realni dio kompleksnog broja, a broj y imaginarni dio kompleksnog broja z. Pišemo:

$$\operatorname{Re}(z) = x \quad , \quad \operatorname{Im}(z) = y.$$

Zapis $z = x + y \cdot i$ zovemo algebarski ili standardni prikaz kompleksnog broja.
Odredimo skup točaka z kompleksne ravnine za koje vrijedi

$$\operatorname{Re}(z+1) = \operatorname{Im}(z-i).$$

Neka je

$$z = x + y \cdot i.$$

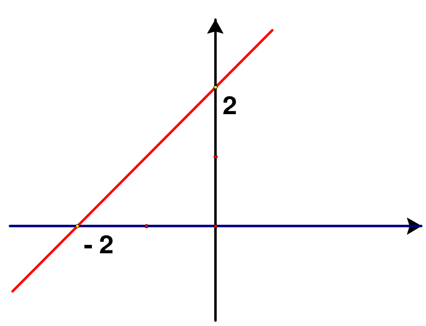
Tada je

$$\left.\begin{aligned} z + 1 &= x + y \cdot i + 1 = (x+1) + y \cdot i \\ z - i &= x + y \cdot i - i = x + (y-1) \cdot i \end{aligned}\right\}.$$

Zadanu jednakost možemo zapisati u obliku

$$\operatorname{Re}(z+1) = \operatorname{Im}(z-i) \Rightarrow \operatorname{Re}((x+1) + y \cdot i) = \operatorname{Im}(x + (y-1) \cdot i) \Rightarrow x + 1 = y - 1 \Rightarrow y = x + 2.$$

Odavde vidimo da koordinate x i y zadovoljavaju jednadžbu pravca y = x + 2.



Vježba 115

Odredi skup točaka z kompleksne ravnine za koje vrijedi $\operatorname{Re}(z+2) = \operatorname{Im}(z - 2 \cdot i)$.

Rezultat: $y = x + 4.$

Zadatak 116 (Davor, gimnazija)

Koliko iznosi realni dio kompleksnog broja $\dfrac{2-i}{2+i} - \dfrac{2+i}{2-i}$?

Rješenje 116

Ponovimo!

Kompleksan broj je broj oblika

$$z = x + y \cdot i,$$

gdje su x i y realni brojevi, a i imaginarna jedinica. Broj x zove se realni dio kompleksnog broja, a broj y imaginarni dio kompleksnog broja z. Pišemo:

$$\mathrm{Re}(z) = x \quad , \quad \mathrm{Im}(z) = y.$$

Zapis $z = x + y \cdot i$ zovemo algebarski ili standardni prikaz kompleksnog broja.

$$\frac{a}{b} - \frac{c}{d} = \frac{a \cdot d - b \cdot c}{b \cdot d} \quad , \quad (a-b)^2 = a^2 - 2 \cdot a \cdot b + b^2 \quad , \quad (a+b)^2 = a^2 + 2 \cdot a \cdot b + b^2.$$

$$i^2 = -1 \quad , \quad (a - b \cdot i) \cdot (a + b \cdot i) = a^2 - b^2 \cdot i^2 = a^2 + b^2.$$

1.inačica

$$\frac{2-i}{2+i} - \frac{2+i}{2-i} = \frac{2-i}{2+i} \cdot \frac{2-i}{2-i} - \frac{2+i}{2-i} \cdot \frac{2+i}{2+i} = \frac{(2-i)^2}{(2+i)\cdot(2-i)} - \frac{(2+i)^2}{(2-i)\cdot(2+i)} =$$

$$= \frac{4 - 4 \cdot i + i^2}{2^2 - i^2} - \frac{4 + 4 \cdot i + i^2}{2^2 - i^2} = \frac{4 - 4 \cdot i - 1}{4 - (-1)} - \frac{4 + 4 \cdot i - 1}{4 - (-1)} = \frac{3 - 4 \cdot i}{4+1} - \frac{3 + 4 \cdot i}{4+1} = \frac{3 - 4 \cdot i}{5} - \frac{3 + 4 \cdot i}{5} =$$

$$= \frac{3 - 4 \cdot i - (3 + 4 \cdot i)}{5} = \frac{3 - 4 \cdot i - 3 - 4 \cdot i}{5} = \frac{3 - 4 \cdot i - 3 - 4 \cdot i}{5} = -\frac{8 \cdot i}{5} = 0 - \frac{8}{5} \cdot i.$$

Realni dio kompleksnog broja iznosi:

$$\frac{2-i}{2+i} - \frac{2+i}{2-i} = 0 - \frac{8}{5} \cdot i \Rightarrow \left.\begin{array}{l} \text{Realni dio} = 0 \\ \text{Imaginarni dio} = -\frac{8}{5} \end{array}\right\}.$$

2.inačica

$$\frac{2-i}{2+i} - \frac{2+i}{2-i} = \frac{(2-i)^2 - (2+i)^2}{(2+i)\cdot(2-i)} = \frac{2^2 - 4 \cdot i + i^2 - \left(2^2 + 4 \cdot i + i^2\right)}{2^2 - i^2} = \frac{2^2 - 4 \cdot i + i^2 - 2^2 - 4 \cdot i - i^2}{4 - (-1)} =$$

$$= \frac{2^2 - 4 \cdot i + i^2 - 2^2 - 4 \cdot i - i^2}{4+1} = \frac{-8 \cdot i}{5} = 0 - \frac{8}{5} \cdot i.$$

Realni dio kompleksnog broja iznosi:

$$\frac{2-i}{2+i} - \frac{2+i}{2-i} = 0 - \frac{8}{5} \cdot i \Rightarrow \left.\begin{array}{l} \text{Realni dio} = 0 \\ \text{Imaginarni dio} = -\frac{8}{5} \end{array}\right\}.$$

Vježba 116

Koliko iznosi imaginarni dio kompleksnog broja $\frac{2-i}{2+i} - \frac{2+i}{2-i}$?

Rezultat: $-\frac{8}{5}.$

Zadatak 117 (Davor, gimnazija)

Ako je $\frac{x+2}{3+2 \cdot i} - \frac{y+3}{3-2 \cdot i} = 1, \; (x, \; y \in R)$, kolika je vrijednost izraza $5 \cdot x - y$?

Rješenje 117

Ponovimo!

Kompleksan broj je broj oblika

$$z = x + y \cdot i,$$

gdje su x i y realni brojevi, a i imaginarna jedinica. Broj x zove se realni dio kompleksnog broja, a broj y imaginarni dio kompleksnog broja z. Pišemo:

$$\mathrm{Re}(z) = x \quad , \quad \mathrm{Im}(z) = y.$$

Zapis $z = x + y \cdot i$ zovemo algebarski ili standardni prikaz kompleksnog broja.

$$\frac{a}{b} - \frac{c}{d} = \frac{a \cdot d - b \cdot c}{b \cdot d} \quad , \quad (a-b)^2 = a^2 - 2 \cdot a \cdot b + b^2 \quad , \quad (a+b)^2 = a^2 + 2 \cdot a \cdot b + b^2.$$

$$\frac{a}{n} + \frac{b}{n} = \frac{a+b}{n} \quad , \quad i^2 = -1 \quad , \quad (a - b \cdot i) \cdot (a + b \cdot i) = a^2 - b^2 \cdot i^2 = a^2 + b^2.$$

Ponovimo definiciju jednakosti kompleksnih brojeva. Kada su dva kompleksna broja jednaka?

Dva su kompleksna broja jednaka ako i samo ako su im međusobno jednaki realni dijelovi i međusobno jednaki imaginarni dijelovi, tj.

$$a + bi = c + di \quad \Leftrightarrow \quad a = c \, , \, b = d.$$

1.inačica

$$\frac{x+2}{3+2 \cdot i} - \frac{y+3}{3-2 \cdot i} = 1 \Rightarrow \frac{x+2}{3+2 \cdot i} - \frac{y+3}{3-2 \cdot i} = 1 \ / \cdot (3+2 \cdot i) \cdot (3-2 \cdot i) \Rightarrow$$

$$\Rightarrow (x+2) \cdot (3-2 \cdot i) - (y+3) \cdot (3+2 \cdot i) = (3+2 \cdot i) \cdot (3-2 \cdot i) \Rightarrow$$

$$\Rightarrow 3 \cdot x - 2 \cdot x \cdot i + 6 - 4 \cdot i - (3 \cdot y + 2 \cdot y \cdot i + 9 + 6 \cdot i) = 3^2 - (2 \cdot i)^2 \Rightarrow$$

$$\Rightarrow 3 \cdot x - 2 \cdot x \cdot i + 6 - 4 \cdot i - 3 \cdot y - 2 \cdot y \cdot i - 9 - 6 \cdot i = 3^2 - 2^2 \cdot i^2 \Rightarrow$$

$$\Rightarrow 3 \cdot x - 3 \cdot y + 6 - 9 - 2 \cdot x \cdot i - 4 \cdot i - 2 \cdot y \cdot i - 6 \cdot i = 9 - 4 \cdot (-1) \Rightarrow$$

$$\Rightarrow 3 \cdot x - 3 \cdot y - 3 - 2 \cdot x \cdot i - 2 \cdot y \cdot i - 10 \cdot i = 9 + 4 \Rightarrow 3 \cdot x - 3 \cdot y - 3 + (-2 \cdot x - 2 \cdot y - 10) \cdot i = 13 \Rightarrow$$

$$\Rightarrow 3 \cdot x - 3 \cdot y - 3 + (-2 \cdot x - 2 \cdot y - 10) \cdot i = 13 + 0 \cdot i \Rightarrow \begin{bmatrix} \text{jednakost} \\ \text{kompleksnih brojeva} \end{bmatrix} \Rightarrow$$

$$\Rightarrow \left.\begin{matrix} 3 \cdot x - 3 \cdot y - 3 = 13 \\ -2 \cdot x - 2 \cdot y - 10 = 0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} 3 \cdot x - 3 \cdot y = 13 + 3 \\ -2 \cdot x - 2 \cdot y = 10 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} 3 \cdot x - 3 \cdot y = 16 \\ -2 \cdot x - 2 \cdot y = 10 \end{matrix}\right\} \Rightarrow \begin{bmatrix} \text{metoda suprotnih} \\ \text{koeficijenata} \end{bmatrix} \Rightarrow$$

$$\Rightarrow \left.\begin{matrix} 3 \cdot x - 3 \cdot y = 16 \ / \cdot 2 \\ -2 \cdot x - 2 \cdot y = 10 \ / \cdot 3 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} 6 \cdot x - 6 \cdot y = 32 \\ -6 \cdot x - 6 \cdot y = 30 \end{matrix}\right\} \Rightarrow -12 \cdot y = 62 \Rightarrow -12 \cdot y = 62 \ / : (-12) \Rightarrow$$

$$\Rightarrow y = -\frac{62}{12} \Rightarrow y = -\frac{31}{6} \Rightarrow \left.\begin{matrix} 3 \cdot x - 3 \cdot y = 16 \\ y = -\frac{31}{6} \end{matrix}\right\} \Rightarrow 3 \cdot x - 3 \cdot \left(-\frac{31}{6}\right) = 16 \Rightarrow$$

$$\Rightarrow 3 \cdot x + \frac{31}{2} = 16 \Rightarrow 3 \cdot x = 16 - \frac{31}{2} \Rightarrow 3 \cdot x = \frac{32-31}{2} \Rightarrow 3 \cdot x = \frac{1}{2} \ / \cdot \frac{1}{3} \Rightarrow x = \frac{1}{6}.$$

Računamo vrijednost zadanog izraza:

$$5 \cdot x - y = 5 \cdot \frac{1}{6} - \left(-\frac{31}{6}\right) = \frac{5}{6} + \frac{31}{6} = \frac{36}{6} = 6.$$

Kako izbjeći rješavanje sustava jednadžbi?

$$\left.\begin{array}{l}3\cdot x-3\cdot y=16\\-2\cdot x-2\cdot y=10\end{array}\right\}\Rightarrow\left.\begin{array}{l}3\cdot x-3\cdot y=16\\-2\cdot x-2\cdot y=10\ /\cdot(-1)\end{array}\right\}\Rightarrow\left.\begin{array}{l}3\cdot x-3\cdot y=16\\2\cdot x+2\cdot y=-10\end{array}\right\}\Rightarrow\left[\begin{array}{l}\text{zbrojimo}\\\text{jednadžbe}\end{array}\right]\Rightarrow$$

$$\Rightarrow 3\cdot x-3\cdot y+2\cdot x+2\cdot y=16-10\Rightarrow 5\cdot x-y=6.$$

2.inačica

$$\frac{x+2}{3+2\cdot i}-\frac{y+3}{3-2\cdot i}=1\Rightarrow\frac{(x+2)\cdot(3-2\cdot i)-(y+3)\cdot(3+2\cdot i)}{(3+2\cdot i)\cdot(3-2\cdot i)}=1\Rightarrow$$

$$\Rightarrow\frac{3\cdot x-2\cdot x\cdot i+6-4\cdot i-(3\cdot y+2\cdot y\cdot i+9+6\cdot i)}{3^2-(2\cdot i)^2}=1\Rightarrow$$

$$\Rightarrow\frac{3\cdot x-2\cdot x\cdot i+6-4\cdot i-3\cdot y-2\cdot y\cdot i-9-6\cdot i}{3^2-2^2\cdot i^2}=1\Rightarrow$$

$$\Rightarrow\frac{3\cdot x-3\cdot y+6-9-2\cdot x\cdot i-2\cdot y\cdot i-4\cdot i-6\cdot i}{9-4\cdot(-1)}=1\Rightarrow$$

$$\Rightarrow\frac{3\cdot x-3\cdot y-3-2\cdot x\cdot i-2\cdot y\cdot i-10\cdot i}{9+4}=1\Rightarrow\frac{(3\cdot x-3\cdot y-3)+(-2\cdot x-2\cdot y-10)\cdot i}{13}=1\Rightarrow$$

$$\Rightarrow\frac{3\cdot x-3\cdot y-3}{13}+\frac{-2\cdot x-2\cdot y-10}{13}\cdot i=1+0\cdot i\Rightarrow\left[\begin{array}{l}\text{jednakost}\\\text{kompleksnih brojeva}\end{array}\right]\Rightarrow$$

$$\Rightarrow\left.\begin{array}{l}\dfrac{3\cdot x-3\cdot y-3}{13}=1\\\dfrac{-2\cdot x-2\cdot y-10}{13}=0\end{array}\right\}\Rightarrow\left.\begin{array}{l}\dfrac{3\cdot x-3\cdot y-3}{13}=1\ /\cdot 13\\\dfrac{-2\cdot x-2\cdot y-10}{13}=0\ /\cdot 13\end{array}\right\}\Rightarrow$$

$$\Rightarrow\left.\begin{array}{l}3\cdot x-3\cdot y-3=13\\-2\cdot x-2\cdot y-10=0\end{array}\right\}\Rightarrow\left.\begin{array}{l}3\cdot x-3\cdot y=13+3\\-2\cdot x-2\cdot y=10\end{array}\right\}\Rightarrow\left.\begin{array}{l}3\cdot x-3\cdot y=16\\-2\cdot x-2\cdot y=10\end{array}\right\}\Rightarrow\left[\begin{array}{l}\text{metoda suprotnih}\\\text{koeficijenata}\end{array}\right]\Rightarrow$$

$$\Rightarrow\left.\begin{array}{l}3\cdot x-3\cdot y-3=13\\-2\cdot x-2\cdot y-10=0\end{array}\right\}\Rightarrow\left.\begin{array}{l}3\cdot x-3\cdot y=13+3\\-2\cdot x-2\cdot y=10\end{array}\right\}\Rightarrow\left.\begin{array}{l}3\cdot x-3\cdot y=16\\-2\cdot x-2\cdot y=10\end{array}\right\}\Rightarrow\left[\begin{array}{l}\text{metoda suprotnih}\\\text{koeficijenata}\end{array}\right]\Rightarrow$$

$$\Rightarrow\left.\begin{array}{l}3\cdot x-3\cdot y=16\ /\cdot 2\\-2\cdot x-2\cdot y=10\ /\cdot 3\end{array}\right\}\Rightarrow\left.\begin{array}{l}6\cdot x-6\cdot y=32\\-6\cdot x-6\cdot y=30\end{array}\right\}\Rightarrow -12\cdot y=62\Rightarrow -12\cdot y=62\ /:(-12)\Rightarrow$$

$$\Rightarrow y=-\frac{62}{12}\Rightarrow y=-\frac{31}{6}\Rightarrow\left.\begin{array}{l}3\cdot x-3\cdot y=16\\y=-\dfrac{31}{6}\end{array}\right\}\Rightarrow 3\cdot x-3\cdot\left(-\frac{31}{6}\right)=16\Rightarrow$$

$$\Rightarrow 3\cdot x+\frac{31}{2}=16\Rightarrow 3\cdot x=16-\frac{31}{2}\Rightarrow 3\cdot x=\frac{32-31}{2}\Rightarrow 3\cdot x=\frac{1}{2}\ /\cdot\frac{1}{3}\Rightarrow x=\frac{1}{6}.$$

Računamo vrijednost zadanog izraza:

$$5\cdot x-y=5\cdot\frac{1}{6}-\left(-\frac{31}{6}\right)=\frac{5}{6}+\frac{31}{6}=\frac{36}{6}=6.$$

Kako izbjeći rješavanje sustava jednadžbi?

$$\left.\begin{array}{l}3\cdot x-3\cdot y=16\\-2\cdot x-2\cdot y=10\end{array}\right\}\Rightarrow\left.\begin{array}{l}3\cdot x-3\cdot y=16\\-2\cdot x-2\cdot y=10\ /\cdot(-1)\end{array}\right\}\Rightarrow\left.\begin{array}{l}3\cdot x-3\cdot y=16\\2\cdot x+2\cdot y=-10\end{array}\right\}\Rightarrow\left[\begin{array}{l}\text{zbrojimo}\\\text{jednadžbe}\end{array}\right]\Rightarrow$$

$$\Rightarrow 3\cdot x-3\cdot y+2\cdot x+2\cdot y=16-10\Rightarrow 5\cdot x-y=6.$$

Vježba 117

Ako je $\frac{x+2}{3+2\cdot i}-\frac{y+3}{3-2\cdot i}=1, \left(x,\ y\in R\right)$, kolika je vrijednost izraza $10\cdot x-2\cdot y$?

Rezultat: 12.

Zadatak 118 (Medicinarke, medicinska škola)

Uz koji će uvjet kub kompleksnog broja a + b · i biti realan?

Rješenje 118

Ponovimo!

$$(x+y)^3=x^3+3\cdot x^2\cdot y+3\cdot x\cdot y^2+y^3.$$

$$(x\cdot y)^n=x^n\cdot y^n \quad , \quad i^2=-1 \quad , \quad i^3=-i.$$

Zakon distribucije množenja prema zbrajanju.

$$x\cdot(y+z)=x\cdot y+x\cdot z \quad , \quad x\cdot y+x\cdot z=x\cdot(y+z).$$

Kompleksan broj je broj oblika

$$z=x+y\cdot i,$$

gdje su x i y realni brojevi, a i imaginarna jedinica. Broj x zove se realni dio kompleksnog broja, a broj y imaginarni dio kompleksnog broja z. Pišemo:

$$\mathrm{Re}(z)=x \quad , \quad \mathrm{Im}(z)=y.$$

Zapis $z=x+y\cdot i$ zovemo algebarski ili standardni prikaz kompleksnog broja.

Računamo kub kompleksnog broja a + b · i.

$$(a+b\cdot i)^3=a^3+3\cdot a^2\cdot b\cdot i+3\cdot a\cdot(b\cdot i)^2+(b\cdot i)^3=a^3+3\cdot a^2\cdot b\cdot i+3\cdot a\cdot b^2\cdot i^2+b^3\cdot i^3=$$

$$=a^3+3\cdot a^2\cdot b\cdot i+3\cdot a\cdot b^2\cdot(-1)+b^3\cdot(-i)=a^3+3\cdot a^2\cdot b\cdot i-3\cdot a\cdot b^2-b^3\cdot i=$$

$$=a^3-3\cdot a\cdot b^2+3\cdot a^2\cdot b\cdot i-b^3\cdot i=a^3-3\cdot a\cdot b^2+\left(3\cdot a^2\cdot b-b^3\right)\cdot i \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l}\mathrm{Re}(a+b\cdot i)^3=a^3-3\cdot a\cdot b^2\\ \mathrm{Im}(a+b\cdot i)^3=3\cdot a^2\cdot b-b^3\end{array}\right\}.$$

Budući da kub kompleksnog broja mora biti realan, njegov imaginarni dio jednak je nuli.

$$3\cdot a^2\cdot b-b^3=0 \Rightarrow b\cdot\left(3\cdot a^2-b^2\right)=0 \Rightarrow \left.\begin{array}{l}b=0\\ 3\cdot a^2-b^2=0\end{array}\right\} \Rightarrow \left.\begin{array}{l}b=0\\ 3\cdot a^2=b^2\end{array}\right\}.$$

Vježba 118

Uz koji će uvjet kub kompleksnog broja a + b · i biti imaginaran?

Rezultat: $a=0$ ili $a^2=3\cdot b^2$.

Zadatak 119 (Matija, maturant)

Zadani su kompleksni brojevi $z_1=(a+5)\cdot(2-i)$ i $z_2=3-2\cdot b\cdot i$, za $a,\ b\in R$. Odredite b tako da brojevi z_1 i z_2 budu jednaki.

Rješenje 119

Ponovimo!

Ponovimo definiciju jednakosti kompleksnih brojeva. Kada su dva kompleksna broja jednaka?

Dva su kompleksna broja jednaka ako i samo ako su im međusobno jednaki realni dijelovi i međusobno jednaki imaginarni dijelovi, tj.

$$a + b \cdot i = c + d \cdot i \quad \Leftrightarrow \quad a = c\,,\ b = d.$$

$$z_1 = (a+5)\cdot(2-i) \;\Rightarrow\; z_1 = 2\cdot(a+5) - (a+5)\cdot i.$$

Budući da kompleksni brojevi z_1 i z_2 moraju biti jednaki, slijedi:

$$\left.\begin{array}{l} z_1 = 2\cdot(a+5) - (a+5)\cdot i \\ z_2 = 3 - 2\cdot b\cdot i \end{array}\right\} \Rightarrow \left.\begin{array}{l} 2\cdot(a+5) = 3 \\ -(a+5) = -2\cdot b \end{array}\right\} \Rightarrow \left.\begin{array}{l} 2\cdot(a+5) = 3 \;/:2 \\ -(a+5) = -2\cdot b \;/\cdot(-1) \end{array}\right\} \Rightarrow \left.\begin{array}{l} a+5 = \frac{3}{2} \\ a+5 = 2\cdot b \end{array}\right\} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} \text{metoda} \\ \text{komparacije} \end{bmatrix} \Rightarrow 2\cdot b = \frac{3}{2} \;\Rightarrow\; 2\cdot b = \frac{3}{2} \;/:2 \;\Rightarrow\; b = \frac{3}{4}.$$

Vježba 119

Zadani su kompleksni brojevi $z_1 = (a+5)\cdot(2-i)$ i $z_2 = 3 - 2\cdot b\cdot i$, za $a,\ b \in R$. Odredite a tako da brojevi z_1 i z_2 budu jednaki.

Rezultat: $a = -\frac{7}{2}.$

Zadatak 120 (Nina, gimnazija)

Odredite skup točaka u ravnini zadanih relacijom $|z| = \mathrm{Re}(z) + 1.$

Rješenje 120

Ponovimo!

Standardni ili algebarski oblik kompleksnog broja je oblika

$$z = x + y\cdot i,$$

gdje su x i y realni brojevi.

U kompleksnom broju z = x + y · i realni broj x jest njegov realni dio, x = Re (z).

$$z = x + y\cdot i \;\Rightarrow\; |z| = \sqrt{x^2 + y^2}.$$

$$\left(\sqrt{a}\right)^2 = a \quad , \quad (a+b)^2 = a^2 + 2\cdot a\cdot b + b^2.$$

Neka je z = x + y · i. Tada vrijedi:

$$|z| = \mathrm{Re}(z) + 1 \;\Rightarrow\; |x + y\cdot i| = x + 1 \;\Rightarrow\; \sqrt{x^2 + y^2} = x + 1 \;\Rightarrow\; \sqrt{x^2 + y^2} = x + 1 \;/^2 \;\Rightarrow$$

$$\Rightarrow \left(\sqrt{x^2 + y^2}\right)^2 = (x+1)^2 \;\Rightarrow\; x^2 + y^2 = x^2 + 2\cdot x + 1 \;\Rightarrow\; x^2 + y^2 = x^2 + 2\cdot x + 1 \;\Rightarrow$$

$$\Rightarrow\; y^2 = 2\cdot x + 1.$$

Traženi skup točaka je parabola.

Vježba 120

Odredite skup točaka u ravnini zadanih relacijom $|z| = \mathrm{Re}(z) - 1.$

Rezultat: Parabola: $y^2 = -2\cdot x + 1.$